Field theory and weak Euler-Lagrange equation for classical particle-field systems

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It is commonly believed as a fundamental principle that energy-momentum conservation of a physical system is the result of space-time symmetry. However, for classical particle-field systems, e.g., charged particles interacting through self-consistent electromagnetic or electrostatic fields, such a connection has only been cautiously suggested. It has not been formally established. The difficulty is due to the fact that the dynamics of particles and the electromagnetic fields reside on different manifolds. We show how to overcome this difficulty and establish the connection by generalizing the Euler-Lagrange equation, the central component of a field theory, to a so-called *weak* form. The weak Euler-Lagrange equation induces a new type of flux, called the weak Euler-Lagrange current, which enters conservation laws. Using field theory together with the weak Euler-Lagrange equation developed here, energy-momentum conservation laws that are difficult to find otherwise can be systematically derived from the underlying space-time symmetry.

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I. INTRODUCTION

It has been widely accepted as a fundamental principle of physics that energy-momentum conservation of a classical or quantum system is due to the underlying space-time symmetry that the system admits. However, for classical systems with particles and self-generated interacting fields, the connection between energy-momentum conservation and space-time symmetry has only been cautiously suggested [1] and has not been formally established. Examples of such classical particle-field systems include charged particles in an accelerator or a magnetic confinement device interacting through the self-consistent electromagnetic fields [2]. To understand and overcome this difficulty, we need to examine the details of the field theory. In the standard field theory, one writes down a Lagrangian density L, and the associated Euler-Lagrange equation determines the dynamics of the system. When the Euler-Lagrange equation is satisfied, a symmetry condition is equivalent to a conservation law. This is of course the celebrated Noether's theorem [3,4]. It is surprising to find out for classical particle-field systems that the standard Euler-Lagrange equation does not hold anymore. This is because the dynamics of the particles and the electromagnetic fields reside on different manifolds. The electromagnetic fields are defined on the space-time domain, whereas the particle trajectories as a field are only defined on the time-axis. This is why the link between the symmetry and conservation law breaks down for these systems. This unique feature has not been discussed before, and it makes a significant difference in the formulation of the field theory presented here. What we have discovered is that when the standard Euler-Lagrange equation breaks down, the field equations of these systems assume a more general form that can be viewed as a weak Euler-Lagrange equation. It a pleasant surprise to find out that this weak Euler-Lagrange equation can also link symmetries with conservation laws as in the standard field theory, where the regular Euler-Lagrange equation provides the link. The difference is that the weak Euler-Lagrange equation induces a new type of current (unknown previously), called the weak Euler-Lagrange current, in conservation laws, in addition to the Noether current for the standard field theory. For many classical particle-field systems, such as particles interacting through electrostatic potentials [2,5] or attracting Newtonian potentials [6,7], energy-momentum conservation laws are difficult to find. Using the field theory with the weak Euler-Lagrange equation developed here, energy-momentum conservation laws can be systematically derived from the underlying space-time symmetries.

This paper is organized as follows. In Sec. II, classical particle-field systems and the difficulty of establishing the connections between symmetries and conservation laws are introduced. The weak Euler-Lagrange equation and its role in establishing conservation laws are given in Sec. III. The last section summarizes the main results of the paper.

II. CLASSICAL PARTICLE-FIELD SYSTEMS

The classical non-relativistic particle-field system in flat space is governed by the Newton-Maxwell equations

$$\ddot{\boldsymbol{X}}_{sp} = \left(\frac{q}{m}\right)_s \left(\boldsymbol{E} + \frac{1}{c}\dot{\boldsymbol{X}}_{sp}\times\boldsymbol{B}\right), \tag{1}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_{s,p} q_s\delta(\boldsymbol{X}_{sp}-\boldsymbol{x}), \tag{2}$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_{s,p} q_s\dot{\boldsymbol{X}}_{sp}\delta(\boldsymbol{X}_{sp}-\boldsymbol{x}) + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}, \tag{3}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}, \tag{4}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{5}$$

where $\boldsymbol{X}_{sp}(t)$ as a function of time is the trajectory of the pth particle of the s-species, and q_s and m_s are the particle charge and mass, respectively. The electric field $\boldsymbol{E}(\boldsymbol{x},t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ are functions of space-time. Equations (1)–(3) can be expressed equivalently in the form of the Klimontovich-Maxwell (KM) equations [2]

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v}\cdot\frac{\partial F_s}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\frac{\partial F_s}{\partial \boldsymbol{v}} = 0, \quad (6)$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_s q_s \int F_s d_s^3\boldsymbol{v}, \quad (7)$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_s q_s\int F_s\boldsymbol{v}d^3\boldsymbol{v} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}, \quad (8)$$

where $F_s(\boldsymbol{x},\boldsymbol{v},t) = \sum_p \delta(\boldsymbol{X}_{sp}-\boldsymbol{x})\delta(\dot{\boldsymbol{X}}_{sp}-\boldsymbol{v})$ is the Klimontovich distribution function in the phase space $(\boldsymbol{x},\boldsymbol{v})$.

Reduced models are often used in plasma physics. For example, the electrostatic Klimontovich-Poisson (KP) system is given by

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v}\cdot\frac{\partial F_s}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \left(-\nabla\phi + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_0\right)\cdot\frac{\partial F_s}{\partial \boldsymbol{v}} = 0, \quad (9)$$

$$\nabla^2\phi = -4\pi\sum_s q_s\int F_s d^3\boldsymbol{v}, \quad (10)$$

where $\boldsymbol{B}_0(\boldsymbol{x})$ is a background magnetic field produced by steady external currents, and $\boldsymbol{E} = -\nabla\phi$ is the longitudinal electric field. Another well-known reduced model is the Klimontovich-Darwin (KD) system [8–11],

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v}\cdot\frac{\partial F_s}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\frac{\partial F_s}{\partial \boldsymbol{v}} = 0, \quad (11)$$

$$\nabla^2\phi + \nabla\cdot\left(\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right) = -4\pi\sum_s q_s\int F_s d^3\boldsymbol{v}, \quad (12)$$

$$\nabla\times(\nabla\times\boldsymbol{A}) + \frac{1}{c}\frac{\partial\nabla\phi}{\partial t} = \frac{4\pi}{c}\sum_s q_s\int F_s\boldsymbol{v}d^3\boldsymbol{v}, \quad (13)$$

$$\boldsymbol{E} \equiv -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi, \quad \boldsymbol{B}\equiv\nabla\times\boldsymbol{A}. \quad (14)$$

The local energy-momentum conservation laws for the Klimontovich-Maxwell system (6)–(8) is well known [2],

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi} + \sum_{s,p}\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\delta_2\right] + \nabla\cdot\left[\frac{c\boldsymbol{E}\times\boldsymbol{B}}{4\pi} + \sum_{s,p}\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\dot{\boldsymbol{X}}_{sp}\delta_2\right] = 0, \quad (15)$$

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi c} + \sum_{s,p}m_s\dot{\boldsymbol{X}}_{sp}\delta_2\right] + \nabla\cdot\left[\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}I - \frac{\boldsymbol{E}\boldsymbol{E}+\boldsymbol{B}\boldsymbol{B}}{4\pi} + \sum_{s,p}m_s\dot{\boldsymbol{X}}_{sp}\dot{\boldsymbol{X}}_{sp}\delta_2\right] = 0, \quad (16)$$

where we have introduced $\delta_2 \equiv \delta(\boldsymbol{X}_{sp}-\boldsymbol{x})$ to simplify the notation. Through the following identities:

$$\sum_p\frac{\dot{\boldsymbol{X}}_{sp}^2}{2}\delta_2 = \int d^3\boldsymbol{v}F_s\frac{\boldsymbol{v}^2}{2}, \quad \sum_p\frac{\dot{\boldsymbol{X}}_{sp}^2}{2}\dot{\boldsymbol{X}}_{sp}\delta_2 = \int d^3\boldsymbol{v}F_s\frac{\boldsymbol{v}^2}{2}\boldsymbol{v}, \quad (17)$$

$$\sum_p\dot{\boldsymbol{X}}_{sp}\delta_2 = \int d^3\boldsymbol{v}F_s\boldsymbol{v}, \quad \sum_p\dot{\boldsymbol{X}}_{sp}\dot{\boldsymbol{X}}_{sp}\delta_2 = \int d^3\boldsymbol{v}F_s\boldsymbol{v}\boldsymbol{v}, \quad (18)$$

the conservation laws can be expressed equivalently in terms of the distribution function F_s,

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi} + \sum_s\int d^3\boldsymbol{v}F_s m_s\frac{\boldsymbol{v}^2}{2}\right] + \nabla\cdot\left[\frac{c\boldsymbol{E}\times\boldsymbol{B}}{4\pi} + \sum_s\int d^3\boldsymbol{v}F_s m_s\frac{\boldsymbol{v}^2}{2}\boldsymbol{v}\right] = 0, \quad (19)$$

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi c} + \sum_s\int d^3\boldsymbol{v}F_s m_s\boldsymbol{v}\right] + \nabla\cdot\left[\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}I - \frac{\boldsymbol{E}\boldsymbol{E}+\boldsymbol{B}\boldsymbol{B}}{4\pi} + \sum_s\int d^3\boldsymbol{v}F_s m_s\boldsymbol{v}\boldsymbol{v}\right] = 0. \quad (20)$$

For the reduced systems, e.g., the KP system and the KD system, it is also critical to know the exact local energy-momentum conservation laws admitted by the models. In practical applications, such as current drive and heating with lower-hybrid waves [12], and electrostatic drift-wave turbulence, such local energy-momentum conservation laws for the reduced system have profound implications [13–15]. We emphasize that we are looking for the exact conservation laws admitted by the KP and KD systems, which are not exact special cases of the KM system, and should be viewed as independent systems in their own right. For example, we cannot take the exact energy-momentum equations (19) and (20), and approximate $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ to obtain the exact energy-momentum conservation law for the KP system, even though the conservation law obtained this way could be an approximate one for the KP system. The existence of exact local conservation laws is a necessary condition for the models to be theoretically well posed and for the validity of particle simulations based on the KP or KD systems [10].

On the other hand, conservation laws and symmetries are closely related. It is commonly believed that, according to Noether's theorem [3,4], conservation laws can be derived from the symmetries of the corresponding field theories. In standard field theories, this is certainly true, and the symmetry in time for the action is related to energy conservation, and the symmetry in space corresponds to momentum conservation. Therefore, it is reasonable to expect that by analyzing the symmetries of the actions and Lagrangian densities for the reduced systems considered here, we may be able to systematically derive the desired conservation laws. However, it is surprising to find out that for particle-field systems considered here, the field theory works differently. First, let us recall the action and Lagrangian density for the KM system given by Low [16],

$$\mathcal{A}[\phi,\boldsymbol{A},\boldsymbol{X}_{sp}] = \int L_{\mathrm{KM}} d^3\boldsymbol{x}dt, \quad L_{\mathrm{KM}} = L_{\mathrm{KMF}} + L_{\mathrm{KMP}}, \tag{21}$$

$$L_{\mathrm{KMF}} = \left(\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right)^2 \Big/ 8\pi - (\nabla\times\boldsymbol{A})^2/8\pi, \tag{22}$$

$$L_{\mathrm{KMP}} = \sum_{s,p}\left[-q_s\phi + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{A} + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right]\delta_2. \tag{23}$$

It is straightforward to verify that Eqs. (1)–(5) follow from $\delta\mathcal{A}/\delta\boldsymbol{X}_{sp} = 0$, $\delta\mathcal{A}/\delta\phi = 0$, and $\delta\mathcal{A}/\delta\boldsymbol{A} = 0$. For the KP system, the action and Lagrangian density are given by

$$\mathcal{A}[\phi,\boldsymbol{X}_{sp}] = \int L_{\mathrm{KP}} d^3\boldsymbol{x}dt, \quad L_{\mathrm{KP}} = L_{\mathrm{KPF}} + L_{\mathrm{KPP}}, \tag{24}$$

$$L_{\mathrm{KPF}} = (\nabla\phi)^2/8\pi,$$

$$L_{\mathrm{KPP}} = \sum_{s,p}\left[-q_s\phi + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{A}_0 + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right]\delta_2, \tag{25}$$

where $\boldsymbol{A}_0$ is the vector potential for a given external magnetic field $\boldsymbol{B}_0 = \nabla\times\boldsymbol{A}_0$. For the KD system, the action and Lagrangian density for the KM system are

$$\mathcal{A}[\phi,\boldsymbol{A},\boldsymbol{X}_{sp}] = \int L_{\mathrm{KD}} d^3\boldsymbol{x}dt, \quad L_{\mathrm{KD}} = L_{\mathrm{KDF}} + L_{\mathrm{KDP}}, \tag{26}$$

$$L_{\mathrm{KDF}} = \left[\frac{2}{c}\nabla\phi\cdot\frac{\partial \boldsymbol{A}}{\partial t} + (\nabla\phi)^2\right]\Big/ 8\pi - (\nabla\times\boldsymbol{A})^2/8\pi, \tag{27}$$

$$L_{\mathrm{KDP}} = \sum_{s,p}\left[-q_s\phi + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{A} + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right]\delta_2. \tag{28}$$

Based on the spirit of Noether's theorem, we would like to determine whether the local energy-momentum conservation laws can be derived from the symmetries of the corresponding Lagrangian density. It turns out that the answer to this question is not as simple as that in standard field theory. This is because the fields in the present field theory, i.e., $\boldsymbol{X}_{sp}(t)$, $\phi(\boldsymbol{x},t)$, and $\boldsymbol{A}(\boldsymbol{x},t)$ are defined on different domains. The potentials are defined on the space-time domain $(\boldsymbol{x},t)$, whereas the particle trajectory $\boldsymbol{X}_{sp}(t)$ is only defined on the time-axis. This unique feature has not been discussed before, and it makes a significant difference in the formulation of the field theory presented here.

In the next section, we develop the field theory for classical particle-systems with this feature, in particular, the KM system, the KP system, and the KD system. The most distinct characteristic of the field theory presented here is that the field equation for $\boldsymbol{X}_{sp}(t)$ assumes a form we call the weak Euler-Lagrange (EL) equation, which is different from the standard Euler-Lagrange equation. The necessity of using the weak EL equation is mandated by the fact that $\boldsymbol{X}_{sp}(t)$, as a field, is not defined on the entire space-time domain, but only on the time-axis. The weak EL equation with respect to $\boldsymbol{X}_{sp}(t)$ plays an indispensable role in the symmetry analysis and derivation of local conservation laws. For the KP system and KD system, the analysis developed here enables us to determine the desired local conservation laws, which have not been systematically discussed in the literature. For the KM system, where the local energy-momentum conservation laws (19), (20) are known, the present analysis serves the purpose of establishing a connection between the energy-momentum conservation laws and symmetries of the Lagrangian density L_{KM}. Interestingly, such a connection has only been cautiously suggested [1] but not explicitly established previously. This is perhaps not surprising, because the weak EL equation developed here is needed to establish the connection. Due to the space limitation, we present in Sec. III the detailed derivation of the field theory and new conservation laws only for the KP system of a magnetized plasma, and summarize the main results for the KM and the KD systems at the end.

In plasma physics, one often works with the Vlasov-Maxwell (VM) system. Equations (6)–(8) recover the VM equations when two-particle correlations (collisions) become negligibly small as the number of particles becomes increasingly large, while holding total charge and and charge to mass ratio fixed. In the present study, we work with the Klimontovich-Maxwell system, Eqs. (6)–(8) or Eqs. (1)–(3), and pass to the limit of the Vlasov-Maxwell system when necessary under the assumption of negligible collisions. Similarly, the Vlasov-Poisson (VP) and Vlasov-Darwin (VD) systems are regarded as the collisionless limits of the KP and KD systems, respectively.

As a reduced system, the KP (or VP) system describes many important physical processes when the characteristic velocity of the particles or waves are much slower than the speed of light. These include electrostatic waves in plasmas (Langmuir waves) [5], and collective dynamics and excitations in charged particle beams in a frame moving with the beam [2]. The fundamental theory of Landau damping [17] was first developed for the VP (or KP) system. In astrophysics, the VP (or KP) system has also been used to model the collective dynamics of self-gravitating systems with an attractive Newtonian potential [6,7]. Because of these important applications, the VP (or KP) system and its associated Landau damping have also been studied with great interest in the mathematical physics community [18,19].

We also note that while our focus here is on particle-field systems, Eulerian field theories for the VM and VP systems have been developed by Morrison *et al.* [20–23] using a variety of theoretical constructions. In Eulerian theories, the particle distribution in phase space replaces $\boldsymbol{X}_{sp}(t)$ as the field variable.

III. WEAK EULER-LAGRANGE EQUATION, SYMMETRY, AND CONSERVATION LAWS

We begin with Eq. (24) for the KP system, and determine how the action and Lagrangian density vary in response to the field variation $\delta \boldsymbol{X}_{sp}$ and $\delta\phi(\boldsymbol{x},t)$,

$$\delta\mathcal{A} = \int d^3\boldsymbol{x}dt\,\delta\phi E_\phi(L_{\rm KP}) + \sum_{s,p=1}^{N}\int dt\,\delta\boldsymbol{X}_{sp}\cdot\int d^3\boldsymbol{x}\,E_{X_{sp}}(L_{\rm KP}), \tag{29}$$

$$E_\phi(L_{\rm KP}) \equiv \frac{\partial L_{\rm KP}}{\partial\phi} - \frac{D}{Dx^i}\frac{\partial L_{\rm KP}}{\partial\phi_{,i}},$$
$$E_{X_{sp}}(L_{\rm KP}) \equiv \frac{\partial L_{\rm KP}}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\left(\frac{\partial L_{\rm KP}}{\partial\dot{\boldsymbol{X}}_{sp}}\right). \tag{30}$$

In Eq. (29), $\phi_{,i}\equiv\partial\phi/\partial x^i$ and integration by parts has been applied with respect to terms containing $\partial L_{\rm KP}/\partial\phi_{,i}$ and $\partial L_{\rm KP}/\partial\dot{\boldsymbol{X}}_{sp}$. Here, $E_\phi(L_{\rm KP})$ and $E_{X_{sp}}(L_{\rm KP})$ are the Euler operators with respect to ϕ and $\boldsymbol{X}_{sp}$, respectively. For a variable h, Dh/Dx^i and Dh/Dt represent the space-time derivatives when $h=h(\boldsymbol{x},t)$ is considered as a field on the space-time domain. Because $\delta\phi(\boldsymbol{x},t)$ is arbitrary, $\delta\mathcal{A}/\delta\phi=0$ requires the Euler-Lagrange (EL) equation for ϕ to hold, i.e., $E_\phi(L_{\rm KP})=0$, which is indeed the Poisson equation (10), as expected. The field equation for $\boldsymbol{X}_{sp}$ is more interesting. Because $\delta\boldsymbol{X}_{sp}$ is arbitrary only on the time-axis, the condition $\delta\mathcal{A}/\delta\boldsymbol{X}_{sp}=0$ requires only that the integral of $E_{X_{sp}}(L_{\rm KP})$ over space vanish, i.e.,

$$\int d^3\boldsymbol{x}\,E_{X_{sp}}(L_{\rm KP}) = 0. \tag{31}$$

Equation (31) will be called the submanifold Euler-Lagrangian equation because it is defined only on the time-axis after the integrating over the spatial variable. If $\boldsymbol{X}_{sp}$ were a function of the entire space-time domain, then $E_{X_{sp}}(L_{\rm KP})$ would vanish everywhere, as in the case for $\phi(\boldsymbol{x},t)$. In general, we expect that $E_{X_{sp}}(L_{\rm KP})\neq 0$.

We now derive an explicit expression for $E_{X_{sp}}(L_{\rm KP})$. For the first term in $E_{X_{sp}}(L_{\rm KP})$,

$$\frac{\partial L_{\rm KP}}{\partial\boldsymbol{X}_{sp}} = \left(\frac{q_s}{c}\boldsymbol{A}_0\cdot\dot{\boldsymbol{X}}_{sp} - q_s\phi + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\frac{\partial\delta_2}{\partial\boldsymbol{X}_{sp}}$$
$$= \frac{\partial}{\partial\boldsymbol{x}}(H_{sp}-\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{P}_{sp}) + \left(\frac{q_s}{c}\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\cdot\dot{\boldsymbol{X}}_{sp} - q_s\frac{\partial\phi}{\partial\boldsymbol{x}}\right)\delta_2, \tag{32}$$

where the momentum $\boldsymbol{P}_{sp}$ density and Hamiltonian H_{sp} density are defined as

$$\boldsymbol{P}_{sp}(\boldsymbol{x},t) \equiv \frac{\partial L_{\rm KP}}{\partial\dot{\boldsymbol{X}}_{sp}} = \left(m_s\dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c}\boldsymbol{A}_0\right)\delta_2,$$
$$H_{sp}(\boldsymbol{x},t) \equiv \left(q_s\phi + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2. \tag{33}$$

The second term in $E_{X_{sp}}(L_{\rm KP})$ is given by

$$\frac{D}{Dt}\frac{\partial L_{\rm KP}}{\partial\dot{\boldsymbol{X}}_{sp}} = m_s\ddot{\boldsymbol{X}}_{sp}\delta_2 + \left(m_s\dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c}\boldsymbol{A}_0\right)\frac{\partial\delta_2}{\partial t}$$
$$= m_s\ddot{\boldsymbol{X}}_{sp}\delta_2 - \frac{\partial}{\partial\boldsymbol{x}}\cdot(\dot{\boldsymbol{X}}_{sp}\boldsymbol{P}_{sp}) + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\delta_2. \tag{34}$$

Therefore,

$$E_{X_{sp}}(L_{\rm KP}) = \left[\frac{q_s}{c}\left(\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\cdot\dot{\boldsymbol{X}}_{sp} - \dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\right) - q_s\frac{\partial\phi}{\partial\boldsymbol{x}} - m_s\ddot{\boldsymbol{X}}_{sp}\right]\delta_2 + \frac{\partial}{\partial\boldsymbol{x}}(H_{sp}-\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{P}_{sp}) + \frac{\partial}{\partial\boldsymbol{x}}\cdot(\dot{\boldsymbol{X}}_{sp}\boldsymbol{P}_{sp}). \tag{35}$$

Substituting Eq. (35) into the submanifold EL equation (31), we immediately recover Newton's equation for $\boldsymbol{X}_{sp}$, i.e.,

$$\frac{m_s}{q_s}\ddot{\boldsymbol{X}} = -\frac{\partial\phi}{\partial\boldsymbol{x}} + \frac{1}{c}\dot{\boldsymbol{X}}_{sp}\times\boldsymbol{B}_0, \tag{36}$$

which reduces Eq. (35) to

$$E_{X_{sp}}(L_{\rm KP}) \equiv \frac{\partial L_{\rm KP}}{\partial\boldsymbol{X}_{sp}} - \frac{D}{Dt}\left(\frac{\partial L_{\rm KP}}{\partial\dot{\boldsymbol{X}}_{sp}}\right) = \frac{\partial}{\partial\boldsymbol{x}}(H_{sp}-\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{P}_{sp}) + \frac{\partial}{\partial\boldsymbol{x}}\cdot(\dot{\boldsymbol{X}}_{sp}\boldsymbol{P}_{sp}). \tag{37}$$

As expected, $E_{X_{sp}}(L_{\rm KP})\neq 0$. We will refer to Eq. (37) as the weak Euler-Lagrange equation, which is the foundation for the subsequent analysis of the local conservation laws. The qualifier "weak" is used to indicate the fact that only the spatial integral of the Euler derivative $E_{X_{sp}}(L_{\rm KP})$ is zero [see Eq. (31)], in comparison with the standard EL equation, which demands that the Euler derivative vanishes everywhere.

We define a symmetry of the action $\mathcal{A}[\phi,\boldsymbol{X}_{sp}]$ to be a group of transformation

$$(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp}) \mapsto (\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp}) \tag{38}$$

such that

$$\int L_{\rm KP}[\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp}]d^3\boldsymbol{x}dt = \int L_{\rm KP}[\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp}]d^3\tilde{\boldsymbol{x}}d\tilde{t}. \tag{39}$$

If the symmetry is generated by a vector field on the space of $(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp})$,

$$V = \boldsymbol{\xi}\cdot\frac{\partial}{\partial\boldsymbol{x}} + \xi^t\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial\phi} + \boldsymbol{Y}_p\cdot\frac{\partial}{\partial\boldsymbol{X}_{sp}},$$

then the infinitesimal criteria of invariance is given by [4]

$${\rm pr}V(L) + L\,{\rm Div}\xi = 0, \tag{40}$$

where ${\rm Div}\xi$ is the divergence of the vector field

$$\xi = \boldsymbol{\xi}\cdot\frac{\partial}{\partial\boldsymbol{x}} + \xi^t\frac{\partial}{\partial t} \tag{41}$$

on the space-time domain, and ${\rm pr}V$ is the prolongation of the vector field V on $(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp})$. The prolongation ${\rm pr}V$ is a vector field on the jet space, consisting of the space of $(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp})$ and the space of derivatives of $(\phi,\boldsymbol{X}_{sp})$ with respect to $(\boldsymbol{x},t)$. A comprehensive description of this subject can be found in Ref. [4]. Given the symmetry vector field V, the infinitesimal criteria for invariance will generate the desired conservation law corresponding to the symmetry vector field V, after use is made of the EL equation as well as the weak EL equation for the systems in the present study. We first look for the symmetry group that generates local energy conservation. The group of transformation

$$(\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x},t+\epsilon,\phi,\boldsymbol{X}_{sp}),\quad \epsilon\in R \tag{42}$$

is a symmetry of L_{KP}, because L_{KP} does not depend on t explicitly, i.e., $\partial L_{\mathrm{KP}}/\partial t = 0$, which can be written as

$$\frac{DL_{\mathrm{KP}}}{Dt} - \phi_{,t}\frac{\partial L_{\mathrm{KP}}}{\partial \phi} - \phi_{,jt}\frac{\partial L_{\mathrm{KP}}}{\partial \phi_{,j}} - \sum_{s,p}\left(\dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial L_{\mathrm{KP}}}{\partial \boldsymbol{X}_{sp}} + \ddot{\boldsymbol{X}}_{sp}\cdot\frac{\partial L_{\mathrm{KP}}}{\partial \dot{\boldsymbol{X}}_{sp}}\right) = 0. \quad (43)$$

Equation (43) is the special form of Eq. (40) for this symmetry group. From the EL equation for ϕ, i.e., $E_\phi(L_{\mathrm{KP}}) = 0$, we obtain

$$\phi_{,t}\frac{\partial L_{\mathrm{KP}}}{\partial \phi} + \phi_{,jt}\frac{\partial L_{\mathrm{KP}}}{\partial \phi_{,j}} = \frac{D}{Dx^j}\left(\phi_{,t}\frac{\partial L_{\mathrm{KP}}}{\partial \phi_{,j}}\right). \quad (44)$$

The weak EL equation for $\boldsymbol{X}_{sp}$, i.e., Eq. (37), gives

$$\dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial L_{\mathrm{KP}}}{\partial \boldsymbol{X}_{sp}} + \ddot{\boldsymbol{X}}_{sp}\cdot\frac{\partial L_{\mathrm{KP}}}{\partial \dot{\boldsymbol{X}}_{sp}} = \frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\dot{\boldsymbol{X}}_{sp}\left(q_s\phi + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] + \frac{D}{Dt}(\dot{\boldsymbol{X}}_{sp}\cdot\boldsymbol{P}_{sp}). \quad (45)$$

Combining Eqs. (44) and (45), we obtain the first local energy conservation law,

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi} - \sum_{s,p}\left(q_s\phi + \frac{m_s}{2}\dot{\boldsymbol{X}}^2{}_{sp}\right)\delta_2\right] + \frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\frac{-1}{4\pi}\phi_{,t}\nabla\phi - \sum_{s,p}\dot{\boldsymbol{X}}_{sp}\left(q_s\phi + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] = 0. \quad (46)$$

We subtract the identify

$$\frac{1}{4\pi}\frac{\partial}{\partial t}[(\nabla\phi)^2 + \phi\nabla^2\phi] + \frac{1}{4\pi}\frac{\partial}{\partial \boldsymbol{x}}\cdot(-\phi_{,t}\nabla\phi - \phi\nabla\phi_{,t}) = 0 \quad (47)$$

from Eq. (46) to express the energy conservation law in another equivalent form

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi} + \sum_{s,p}\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\delta_2\right] + \frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\sum_{s,p}\dot{\boldsymbol{X}}_{sp}\left(q_s\phi + \frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\right)\delta_2 - \frac{1}{4\pi}\phi\nabla\phi_{,t}\right] = 0. \quad (48)$$

In terms of the distribution function F_s, we obtain

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi} + \sum_s\int F_s\frac{m_s\boldsymbol{v}^2}{2}d^3\boldsymbol{v}\right] + \frac{\partial}{\partial \boldsymbol{x}}\cdot\left(\sum_s\int F_s\frac{m_s\boldsymbol{v}^2}{2}\boldsymbol{v}d^3\boldsymbol{v} + \sum_s q_s\phi\int F_s\boldsymbol{v}d^3\boldsymbol{v} - \frac{1}{4\pi}\phi\nabla\phi_{,t}\right) = 0. \quad (49)$$

We emphasize again that Eq. (49) is the exact energy conservation law admitted by the KP system Eqs. (9) and (10), and it cannot be obtained by replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ in the energy conservation law for the KM sytem (19). The sum of the last two terms in Eq. (49) is the electrostatic Poynting flux of the KP system, first discussed by Similon [24] for an unmagnetized plasma by algebraic manipulation. Its importance for electrostatic particle simulations was addressed by Decyk [25]. Here, it appears naturally as a consequence of the symmetry analysis. We observe that the external $\boldsymbol{B}_0$ does not contribute to the energy flux of the electromagnetic field. To further appreciate the importance of the exact energy conservation law (49), let us consider the well-established technique of current drive and heating of a magnetized plasma using electrostatic lower-hybrid (LH) waves [12], which are adequately described by the VP (or KP) system. In this application, the energy and momentum of the LH waves are converted to those of the particles, and it is of practical importance to know the heating power of a specific LH wave system. However, if we calculated the energy flux of the LH waves from Eq. (19) by replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$, we would find that $\nabla\cdot[\nabla\phi\times\boldsymbol{B}_0] = 0$, i.e., the LH waves do not carry an energy flux. This is obviously erroneous. The typical power of such systems in modern magnetic fusion devices is several megawatts. The correct way to calculate the energy flux of the LH waves is to use Eq. (49) instead. Specifically, the several megawatts of energy carried by the LH waves flow into the plasma through the last two terms in Eq. (49).

Up to now, we have treated the KM and KP systems as independent systems, each of which has its own governing equations, Lagrangian, and conservation laws. On the other hand, it is also correct to treat the KP system as the electrostatic approximation to the KM system. From the perspective of the governing equations, this approximation is equivalent to replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ in the KM system. But this simple procedure does not work for the corresponding conservation laws. What is needed here is a more rigorous procedure to derive the electrostatic approximation that reduces from the KM system to the KP system. After this rigorous procedure is carried out, we find that the correct energy conservation law for the KP system obtained from that of the KM system is actually Eq. (49), instead of that obtained from Eq. (19) by replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$. A similar argument applies to the momentum conservation law for the KP system, i.e., Eq. (57). These derivations are given in detail in the Appendix.

Our next goal is to search for the symmetry that generates the momentum conservation law. In standard field theories, if the Lagrangian density does not depend on $\boldsymbol{x}$ explicitly, then it admits the symmetry of spatial translation, $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \epsilon\boldsymbol{u}$, for a constant vector $\boldsymbol{u}$ and $\epsilon \in R$. Then the usual form of Noether's theorem leads to momentum conservation. This strategy does not work here because L_{KP} depends on $\boldsymbol{x}$ explicitly through $\delta_2 \equiv \delta(\boldsymbol{X}_{sp} - \boldsymbol{x})$ and $\boldsymbol{A}_0(\boldsymbol{x})$. However, if we simultaneously translate both $\boldsymbol{x}$ and $\boldsymbol{X}_{sp}$ by the same amount, then δ_2 is invariant. Thus, we consider the translational transformation

$$(\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x} + \epsilon\boldsymbol{u},t,\phi,\boldsymbol{X}_{sp} + \epsilon\boldsymbol{u}), \quad \epsilon \in R \quad (50)$$

under which $\tilde{\phi}(\tilde{\boldsymbol{x}}) = \phi(\boldsymbol{x}) = \phi(\tilde{\boldsymbol{x}} - \epsilon\boldsymbol{u})$ and $\tilde{\boldsymbol{X}}_{sp}(\tilde{t}) = \boldsymbol{X}_{sp}(t) + \epsilon\boldsymbol{u}$. When $\boldsymbol{A}_0(\boldsymbol{x}) = 0$, we can verify that Eq. (40) is satisfied, and Eq. (50) is indeed a symmetry admitted by L_{KP}. The corresponding vector field is

$$V = \frac{\partial}{\partial \boldsymbol{x}} + \sum_{s,p}\frac{\partial}{\partial \boldsymbol{X}_{sp}}, \quad (51)$$

and V is the only non-vanishing component of $\mathrm{Pr}V$ since it is a constant. The notation $\partial/\partial\boldsymbol{x}$ here represents $\partial/\partial x^i$ for $i = 1,2,3$. In this case, the infinitesimal criterion of invariance in Eq. (40) is

$$\frac{\partial L_{\mathrm{KP}}}{\partial \boldsymbol{x}} + \sum_{s,p} \frac{\partial L}{\partial \boldsymbol{X}_{sp}} = 0. \tag{52}$$

When $\boldsymbol{A}_0(\boldsymbol{x}) \neq 0$, the right-hand side of Eq. (52) will have a source term, and instead we obtain

$$\frac{\partial L_{\mathrm{KP}}}{\partial \boldsymbol{x}} + \sum_{s,p} \frac{\partial L}{\partial \boldsymbol{X}_{sp}} = \sum_{s,p} \dot{\boldsymbol{X}}_{sp} \cdot \frac{\partial \boldsymbol{A}_0}{\partial \boldsymbol{x}} \delta_2. \tag{53}$$

It will be clear shortly that this term represents a part of the momentum input due to the external magnetic field through the Lorentz force. For the first term in Eq. (53), we invoke the EL equation $E_\phi(L_{\mathrm{KP}}) = 0$ to obtain

$$\frac{\partial L_{\mathrm{KP}}}{\partial \boldsymbol{x}} = \frac{D L_{\mathrm{KP}}}{D\boldsymbol{x}} - \frac{D}{Dx^j}\left(\frac{\partial L_{\mathrm{KP}}}{\partial \phi_{,j}} \nabla\phi\right). \tag{54}$$

For the second term in Eq. (53), the weak EL equation for $\boldsymbol{X}_{sp}$ Eq. (37) is applied, which gives

$$\frac{\partial L}{\partial \boldsymbol{X}_{sp}} = \frac{D\boldsymbol{P}_{sp}}{Dt} + \frac{\partial}{\partial \boldsymbol{x}}(H_{sp} - \dot{\boldsymbol{X}}_{sp} \cdot \boldsymbol{P}_{sp}) + \frac{\partial}{\partial \boldsymbol{x}} \cdot (\dot{\boldsymbol{X}}_{sp} \boldsymbol{P}_{sp}). \tag{55}$$

Therefore, the conservation law generated by Eq. (53) is

$$\frac{\partial}{\partial t}\left(\sum_{s,p} m_s \dot{\boldsymbol{X}}_{sp}\delta_2\right) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\sum_{s,p} m_s \dot{\boldsymbol{X}}_{sp}\dot{\boldsymbol{X}}_{sp}\delta_2 + \frac{\boldsymbol{I}}{8\pi}(\nabla\phi)^2 - \frac{1}{4\pi}\nabla\phi\nabla\phi\right] = \sum_{s,p} m_s \frac{\dot{\boldsymbol{X}}_{sp}}{c} \times \boldsymbol{B}_0 \delta_2. \tag{56}$$

Evidently, this is the local momentum conservation. In terms of the distribution function F_s, it can be expressed as

$$\frac{\partial}{\partial t}\left(\sum_s m_s \int F_s \boldsymbol{v} d^3\boldsymbol{v}\right) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\sum_s m_s \int F_s \boldsymbol{v}\boldsymbol{v} d^3\boldsymbol{v} + \frac{\boldsymbol{I}}{8\pi}(\nabla\phi)^2 - \frac{1}{4\pi}\nabla\phi\nabla\phi\right] = \sum_s q_s \left(\int F_s \frac{\boldsymbol{v}}{c} d^3\boldsymbol{v}\right) \times \boldsymbol{B}_0. \tag{57}$$

The first term on the left-hand side of Eqs. (56) and (57) is the rate of variation of the momentum density, the second term is the divergence of the flux, and the term on the right-hand side is the momentum input due to the background magnetic field. Note that the momentum density is purely mechanical, and does not include the electromagnetic momentum density $-\nabla\phi \times \boldsymbol{B}_0/4\pi c$. This is not totally intuitive. This conservation law is the result of the symmetry (50), which is different from the well-known translational symmetry for standard field theory. Because L_{KP} depends on $\boldsymbol{x}$ explicitly through $\delta_2 \equiv \delta(\boldsymbol{X}_{sp} - \boldsymbol{x})$, a translation in $\boldsymbol{x}$ alone is not a symmetry of L_{KP}, even when $\boldsymbol{A}_0(\boldsymbol{x}) = 0$. Instead, the symmetry group (50) simultaneously translates the space $\boldsymbol{x}$ and the field $\boldsymbol{X}_{sp}$ by the same amount.

For the KD system, the weak EL equation for $\boldsymbol{X}_{sp}$ is

$$E_{X_{sp}}(L_{\mathrm{KD}}) \equiv \frac{\partial L_{\mathrm{KD}}}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\frac{\partial L_{\mathrm{KD}}}{\partial \dot{\boldsymbol{X}}_{sp}} = \frac{\partial}{\partial \boldsymbol{x}}\left[\left(-\boldsymbol{A} \cdot \dot{\boldsymbol{X}}_{sp} + \phi - \frac{1}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] + \frac{\partial}{\partial \boldsymbol{x}} \cdot [\dot{\boldsymbol{X}}_{sp}(\dot{\boldsymbol{X}}_{sp} + \boldsymbol{A})\delta_2]. \tag{58}$$

Energy conservation follows from the infinitesimal criterion (40) for the symmetry transformation (42) after the weak EL equation (58) for $\boldsymbol{X}_{sp}$ and the EL equations for ϕ and $\boldsymbol{A}$ are applied, i.e.,

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2 + \boldsymbol{B}^2}{8\pi} + \sum_s \int F_s \frac{m_s \boldsymbol{v}^2}{2} d^3\boldsymbol{v}\right] + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left(\sum_s \int F_s \frac{m_s \boldsymbol{v}^2}{2} \boldsymbol{v} d^3\boldsymbol{v} + \frac{\phi_{,t}\boldsymbol{A}_{,t} + \boldsymbol{E}\times\boldsymbol{B}}{4\pi}\right) = 0.$$

Similarly, the infinitesimal criterion for the symmetry group (50) gives the momentum conservation relation

$$\frac{\partial}{\partial t}\left(\sum_s m_s \int F_s \boldsymbol{v} d^3\boldsymbol{v} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi}\right) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\sum_s m_s \int F_s \boldsymbol{v}\boldsymbol{v} d^3\boldsymbol{v} + \frac{(\nabla\phi)^2 + \boldsymbol{B}^2 + 2\nabla\phi \cdot \boldsymbol{A}_{,t}}{8\pi}\boldsymbol{I} - \frac{\boldsymbol{E}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{B} - \boldsymbol{A}_{,t}\boldsymbol{A}_{,t}}{4\pi}\right] = 0. \tag{59}$$

For the KM system, the weak EL equation for $\boldsymbol{X}_{sp}$ is

$$E_{X_{sp}}(L_{\mathrm{KD}}) \equiv \frac{\partial L_{\mathrm{KD}}}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\frac{\partial L_{\mathrm{KD}}}{\partial \dot{\boldsymbol{X}}_{sp}} = \frac{\partial}{\partial \boldsymbol{x}}\left[\left(-\boldsymbol{A} \cdot \dot{\boldsymbol{X}}_{sp} + \phi - \frac{1}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] + \frac{\partial}{\partial \boldsymbol{x}} \cdot [\dot{\boldsymbol{X}}_{sp}(\dot{\boldsymbol{X}}_{sp} + \boldsymbol{A})\delta_2]. \tag{60}$$

The symmetry groups (42) and (50) give the energy and momentum conservation laws (19) and (20) after the weak EL equation for $\boldsymbol{X}_{sp}$ and EL equations for ϕ and $\boldsymbol{A}$ are applied.

IV. SUMMARY AND CONCLUSIONS

In summary, a close examination of the field theory for classical particle-field systems reveals that the particle field $\boldsymbol{X}_{sp}$ and the electromagnetic field reside on different manifolds. This unique feature is fount to imply that $E_{X_{sp}}(L)$, the Euler derivative of the Lagrangian density L with respect to particle's trajectory $\boldsymbol{X}_{sp}$, does not vanish on the space-time manifold, which is surprisingly different from the standard field theory. In fact,

$$E_{X_{sp}}(L) \equiv \frac{\partial L}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{X}}_{sp}}\right) = \frac{\partial}{\partial \boldsymbol{x}} \cdot \boldsymbol{T}, \tag{61}$$

for some non-vanishing tensor $\boldsymbol{T}$. Equation (61) is what we call the weak Euler-Lagrange equation, and it is the most essential component in establishing the connection between energy-momentum conservation and space-time symmetry for classical particle-field systems. In fact, the energy-momentum conservation law follows from the infinitesimal criterion of the space-time system, after the weak Euler-Lagrange equation is applied. The non-vanishing tensor $\boldsymbol{T}$ is a new type of flux called the weak Euler-Lagrange current that enters the conservation laws. For the Klimontovich-Maxwell (or Vlasov-Maxwell) system, this theoretical construction explicitly links the well-known energy-momentum conservation law with the space-time symmetry, which was only cautiously suggested previously. For reduced systems, such as the Klimontovich-Poisson (or Vlasov-Poisson) system and the Klimontovich-Darwin (Vlasov-Darwin) system, this theoretical construction enable us to start from fundamental symmetry properties to systematically derive the energy-momentum conservation laws, which are difficult to determine otherwise.

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APPENDIX

In Sec. III, we have treated the Klimontovich-Maxwell (KM) and Klimontovich-Poisson (KP) systems as independent systems, each of which has its own governing equations, Lagrangian, and conservation laws. The Vlasov-Poisson (VP) and Vlasov-Darwin (VD) systems are regarded as the collisionless limits of the KP and KD systems, respectively.

On the other hand, it is also correct to treat the KP system as the electrostatic approximation to the KM system. From the perspective of the governing equations, this approximation is equivalent to replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ in the KM system. But this simple procedure does not work for the corresponding conservation laws. In this section, we present a more rigorous procedure to carry out the electrostatic approximation that passes from the KM system to the KP system. After this rigorous procedure is carried out, we find that the correct energy conservation law for the KP system obtained from that of the KM system is actually Eq. (49), instead of that obtained from Eq. (19) by replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$.

The electrostatic approximation applies when the characteristic velocity of the particles v and phase velocity of the waves ω/k is much slower than the speed of light, i.e., when $v/c \sim \omega/ck \sim \epsilon \ll 1$. When this condition is satisfied, it turns out that the KM (or VM) system admits solutions with the following ordering:

$$\boldsymbol{E}_l = \boldsymbol{E}_l^{(0)} + \epsilon \boldsymbol{E}_l^{(1)} + \epsilon^2 \boldsymbol{E}_l^{(2)} + O(\epsilon^3), \tag{A1}$$

$$\boldsymbol{E}_t = \epsilon^2 \boldsymbol{E}_t^{(2)} + O(\epsilon^3), \tag{A2}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \epsilon \boldsymbol{B}^{(1)} + \epsilon^2 \boldsymbol{B}^{(2)} + O(\epsilon^3), \tag{A3}$$

$$F_s = F_s^{(0)} + \epsilon F_s^{(1)} + \epsilon^2 F_s^{(2)} + O(\epsilon^3), \tag{A4}$$

where $\boldsymbol{E}_l$ and $\boldsymbol{E}_t$ are the longitudinal and transverse components of the electric field, respectively, and $\boldsymbol{B}_0$ is the externally applied magnetic field with $\nabla\times\boldsymbol{B}_0 = 0$ inside the plasma. The superscripts "(0)", "(1)", and "(2)" represent the orders ϵ^0, ϵ^1, and ϵ^2. To the leading order in ϵ, i.e., $O(\epsilon^0)$, the KM system is

$$\frac{\partial F_s^{(0)}}{\partial t} + \boldsymbol{v}\cdot\frac{\partial F_s^{(0)}}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \times\left(-\nabla\phi + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_0\right)\cdot\frac{\partial F_s^{(0)}}{\partial \boldsymbol{v}} = 0, \tag{A5}$$

$$\nabla^2\phi = -4\pi\sum_s q_s \int F_s^{(0)} d\boldsymbol{v}, \tag{A6}$$

$$\boldsymbol{E}_l^{(0)} \equiv -\nabla\phi, \tag{A7}$$

which is indeed the KP system. Higher-order equations can be derived in a straightforward manner.

For present purposes, we only need the first-order equation for the first-order magnetic field $\boldsymbol{B}^{(1)}$,

$$\nabla\times\boldsymbol{B}^{(1)} = \frac{4\pi}{c}\sum_s q_s \int v F_s^{(0)} d\boldsymbol{v} + \frac{1}{c}\frac{\partial \boldsymbol{E}_l^{(0)}}{\partial t}. \tag{A8}$$

Note that $\boldsymbol{B}^{(1)}$ is determined by the leading-order fields $F_s^{(0)}$ and $\boldsymbol{E}_l^{(0)}$ due to the fact that $v/c \sim \omega/ck \sim \epsilon \ll 1$. Even though $\boldsymbol{B}^{(1)}$ does not enter the governing equations for the KP system, i.e., Eqs. (A5) and (A6), it will enter the leading order energy conservation law for the KP system. Starting from the exact energy conservation law for the KM system, i.e., Eq. (19), we retain all the leading-order terms to obtain

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi} + \sum_s \int F_s^{(0)}\frac{m_s \boldsymbol{v}^2}{2} d^3\boldsymbol{v}\right] + \nabla\cdot\left[\frac{-c\nabla\phi\times\boldsymbol{B}^{(1)}}{4\pi} + \sum_s \int F_s^{(0)}\frac{m_s \boldsymbol{v}^2}{2}\boldsymbol{v} d^3\boldsymbol{v}\right] = 0. \tag{A9}$$

The term $-c\nabla\phi\times\boldsymbol{B}^{(1)}/4\pi$ is the Poynting flux due to the leading-order longitudinal electric field $\boldsymbol{E}_l^{(0)}$ and the first-order magnetic field $\boldsymbol{B}^{(1)}$, and it must be included in the leading-order energy equation, because $ck/\omega \sim 1/\epsilon$ raises the order of this term by one. Since $\boldsymbol{B}^{(1)}$ is uniquely determined by $F_s^{(0)}$ and $\boldsymbol{E}_l^{(0)}$ through Eq. (A8), the Poynting flux term can be expressed as

$$\begin{aligned}\nabla\cdot\left[\frac{-c\nabla\phi\times\boldsymbol{B}^{(1)}}{4\pi}\right] &= \frac{c}{4\pi}\nabla\phi\cdot\nabla\times\boldsymbol{B}^{(1)} \\ &= \left[\sum_s q_s \int v F_s^{(0)} d\boldsymbol{v} - \frac{1}{4\pi}\nabla\phi_{,t}\right]\cdot\nabla\phi \\ &= \nabla\cdot\left(\sum_s q_s\phi \int F_s^{(0)}\boldsymbol{v} d^3\boldsymbol{v} - \frac{1}{4\pi}\phi\nabla\phi_{,t}\right),\end{aligned} \tag{A10}$$

where the continuity equation derived from Eq. (A5) has been used. Finally, the leading-order energy conservation equation is

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi}+\sum_s\int F_s^{(0)}\frac{m_s\boldsymbol{v}^2}{2}d^3\boldsymbol{v}\right] +\frac{\partial}{\partial \boldsymbol{x}}\cdot\left(\sum_s\int F_s^{(0)}\frac{m_s\boldsymbol{v}^2}{2}\boldsymbol{v}d^3\boldsymbol{v} +\sum_s q_s\phi\int F_s^{(0)}\boldsymbol{v}d^3\boldsymbol{v}-\frac{1}{4\pi}\phi\nabla\phi_{,t}\right)=0, \tag{A11}$$

which is identical to Eq. (49), if $F_s^{(0)}$ is identified with F_s. This demonstrates that the energy conservation derived from the field theoretical approach for the KP system is not only more rigorous in mathematical treatment, but also more correct in physics content than the simple approach of replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ in the energy conservation law for the KM system. A similar argument applies to the momentum conservation law for the KP system, i.e., Eq. (57).

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