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Novel Hamiltonian method for collective dynamics analysis of an intense charged particle beam propagating through a periodic focusing quadrupole lattice[a)]

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Identifying regimes for quiescent propagation of intense beams over long distances has been a major challenge in accelerator research. In particular, the development of systematic theoretical approaches that are able to treat self-consistently the applied oscillating force and the nonlinear self-field force of the beam particles simultaneously has been a major challenge of modern beam physics. In this paper, the recently developed Hamiltonian averaging technique [E. A. Startsev, R. C. Davidson, and M. Dorf, Phys. Rev. ST Accel. Beams **13**, 064402 (2010)] which incorporates both the applied periodic focusing force and the self-field force of the beam particles, is generalized to the case of time-dependent beam distributions. The new formulation allows not only a determination of quasi-equilibrium solutions of the non-linear Vlasov-Poison system of equations but also a detailed study of their stability properties. The corrections to the well-known "smooth-focusing" approximation are derived, and the results are applied to a matched beam with thermal equilibrium distribution function. It is shown that the corrections remain small even for moderate values of the vacuum phase advance σ_v. Nonetheless, because the corrections to the average self-field potential are non-axisymmetric, the stability properties of the different beam quasi-equilibria can change significantly. © *2011 American Institute of Physics*. [doi:10.1063/1.3589441]

I. INTRODUCTION

There is growing interest in studying the detailed equilibrium and stability properties of intense charged particle beams[1] for applications to high energy and nuclear physics, high energy density physics research using intense particle beams, and heavy ion beams for inertial fusion energy and warm dense matter applications, etc.[2,3] In many of these applications, intense charged particle beams have to be transported over long distances through a focusing channel, which provides transverse particle confinement. In a quadrupole focusing channel, the beam particles experience a transverse linear focusing-defocusing (FODO) force, which is a periodic function of time in the beam frame. This oscillating force provides the necessary focusing only in an average sense.[4–8] For intense charged particle beams, this average focusing force must be strong enough to overcome both the thermal and space-charge defocusing of the beam particles.

Identifying regimes for quiescent beam propagation has been one of the main challenges of accelerator research.[9–19] In particular, the development of systematic approaches that are able to treat self-consistently the applied periodic focusing force and the self-field force of the beam particles simultaneously is very important.[20–24] Several recent investigations[25–27] have used standard Hamiltonian perturbative methods.[28–32] With these methods, one searches for the generating function that relates the old set of canonical phase-space variables to the new canonical phase-space variables. The new canonical variables are chosen to have a Hamiltonian that is independent of time. In the standard approach, the generating function is a function of the mixed set of variables (old and new). This makes the perturbative analysis somewhat cumbersome. Recently, we have developed a powerful Hamiltonian technique which avoids many of the problems encountered in previous formulations.[1] An advantage of this new approach is that instead of using a generating function which is a function of the mixed set of variables, we work with functions that depend exclusively on a new non-oscillating set of variables from the outset. This significantly simplifies the analysis and allows the development of an iterative procedure that makes no reference to the generating function in its final form. This approach also provides a more consistent ordering of all relevant quantities in the derivation of the canonical transformation. In this new ordering, all quantifies are expanded in the small parameter $\bar{\epsilon} \sim \sigma_v/2\pi$, which is the square of the small parameter used by previous authors in Refs. 25–27. As a result, the third-order expansion used in the present analysis is equivalent to a sixth-order expansion used in previously developed methods.

For intense charged particle beams, it is important to take special care in determining the self-field potential. For very intense beams, the average defocussing self-field force is almost in balance with the applied external focusing force. Therefore, even small corrections to the average self-fields may play a significant role in determining the intense beam dynamics. In many applications of intense charged particle beams, it is important to transport as much beam current through a given cross-section as possible. In accelerators and beam transport systems, the transverse dimensions of the beam are limited by the dimensions of the surrounding conducting

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boundary. Since the cost of the transport system is directly related to the size of the surface limiting the beam transversely, it is important to know the effects of the conducting boundary on the beam dynamics when designing such systems. In our approach, we work directly with the Green's function for Poisson's equation which simplifies considerably the calculation of the averaged self-field potential due to the oscillatory charges induced on the conducting boundary.

The organization of this paper is as follows. In Sec. II, the equations describing the nonlinear dynamics of an intense charged particle beam propagating through a quadrupole focusing channel are summarized. The dynamical quantities are normalized, and the small expansion parameter ϵ is identified. The perturbative Hamiltonian transformation method developed in Ref. 1 and its generalization to intense charged particle beams with time-dependent average distribution function are briefly discussed, and the canonical transformation for arbitrary quadrupole focusing lattice correct to second order in the small parameter $\bar{\epsilon} \sim \sigma_v/2\pi$ are presented in Sec. III. Expressions for the average self-field potential for a matched beam with thermal equilibrium distribution are derived and analyzed in Sec. IV. Finally, the key results and conclusions are summarized in Sec. V.

II. THEORETICAL MODEL

The transverse dynamics of a coasting intense charged particle beam inside an alternating-gradient focussing channel with externally applied, transverse focusing force with components $F^{\alpha}_{foc} = -\kappa(s)\eta^{\alpha}x^{\alpha}$, where $\kappa(s)$ is the focusing field strength, $\eta^1 = 1$, and $\eta^2 = -1$, can be described by the nonlinear Vlasov-Poisson system of equations for the beam distribution function $f\,(x,\,p,\,s)$ and the normalized self-field potential $\Psi(x,s)$. Here, $s = v_b t$ is the longitudinal coordinate, where $v_b = const.$ is the directed beam velocity. Moreover, we use an index notation where $(x,y) \equiv (x^1,x^2)$ and $(p_x,p_y) \equiv (p^1,p^2)$. For simplicity, we also suppress variable indices inside function definitions, i.e., we employ the notation $f(x^1,x^2,p^1,p^2,s) \equiv f(x,p,s)$ and $\Psi(x^1,x^2,s) \equiv \Psi(x,s)$. It is convenient to introduce the dimensionless re-normalized variables $\bar{x} = x/a$, $\bar{s} = s/S$, $\bar{\kappa}(\bar{s}) = \kappa(s)/\kappa_0$, $\bar{p} = p/(a\kappa_0 S)$, and $\bar{f} = (f/N)a^4(\kappa_0 S)^2$, where S is the period of the applied focusing lattice, a is the characteristic transverse beam dimension, and κ_0 is the characteristic value of the lattice function $\kappa(s)$.

Dropping the bar notation over the normalized variables, the distribution function $f\,(x,\,p,\,s)$ satisfies the nonlinear Vlasov equation[6]

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \sum_{\alpha=1}^{2}\frac{dx^{\alpha}}{ds}\frac{\partial f}{\partial x^{\alpha}} + \sum_{\alpha=1}^{2}\frac{dp^{\alpha}}{ds}\frac{\partial f}{\partial p^{\alpha}} = 0, \tag{1}$$

where

$$\frac{dx^{\alpha}}{ds} = \frac{\partial H}{\partial p^{\alpha}}, \quad \frac{dp^{\alpha}}{ds} = -\frac{\partial H}{\partial x^{\alpha}} \tag{2}$$

are the particle equations of motion. The Hamiltonian $H\,(x,\,p,\,s)$ is defined by

$$H(x,p,s) = \kappa(s)\frac{[\eta^{\alpha}x^{\alpha}x^{\alpha}]}{2} + \epsilon\left\{\frac{[p^{\alpha}p^{\alpha}]}{2} + + \int L(x,x')f(x',p',s)Dx'Dp'\right\}. \tag{3}$$

For simplicity, we adopt a square-bracket notation for summations, e.g., $[x^{\alpha}x^{\alpha}] \equiv \sum_{\alpha=1}^{2} x^{\alpha}x^{\alpha}$. Moreover, for multi-dimensional integrals, we adopt the notation $\int dx_1 dx_2 Z = \int DxZ$. In Eq. (3), ϵ is defined by $\epsilon \equiv S^2\kappa_0$, and the Green's function $L(x,x')$ satisfies the equation

$$\left[\frac{\partial}{\partial x^{\alpha}}\frac{\partial}{\partial x^{\alpha}}\right]L(x,x') = -s_b\delta(x-x') \tag{4}$$

Here, $s_b = 2\pi A/(\kappa_0 S)^2 a^2 = (4\pi q^2 N/m_b a^2\gamma_b^3)/(\kappa_0 S v_b)^2$ is a dimensionless measure of the beam space-charge intensity. For a beam transversely confined by the external focusing lattice, the characteristic maximum value of the normalized intensity s_b is $(s_b)^{max} \sim 1$.[6] In Eq. (3), the function f is normalized according to $\int DxDpf = 1$.

III. PERTURBATIVE HAMILTONIAN TRANSFORMATION METHOD

In our recent study,[1] we studied the matched quasi-equilibrium solutions to the Vlasov equation (1) by searching for a time-dependent canonical transformation of the form[6,25–27] $(x^{\alpha},p^{\alpha},H,s) \rightarrow (Q^{\alpha},P^{\alpha},K,s)$, where

$$x^{\alpha} = x^{\alpha}(Q,P,s), \quad p^{\alpha} = p^{\alpha}(Q,P,s), \tag{5}$$

with *time-independent* transformed Hamiltonian $K\;(Q,\,P)$. Here, we extend this procedure to the case when the transformed Hamiltonian $K(Q,P,\epsilon s,\epsilon^2 s,\ldots)$ is allowed to depend slowly on time.

For every canonical transformation, there is a function $\tilde{S}$ that satisfies the differential relation[28–31]

$$[p^{\alpha}dx^{\alpha}] - Hds = d\tilde{S} + [P^{\alpha}dQ^{\alpha}] - Kds. \tag{6}$$

To develop an iterative series, we have introduced the new functions[1] $U\;(Q,\,P,\,s)$ and $p_0\;(Q,\,P,\,s)$ related to generating function $\tilde{S}$ by $\tilde{S} = U + [p_0(Q,P,s)^{\alpha}(x-Q)^{\alpha}]$. The relationships between the old and new set of phase-space coordinates are obtained from Eq. (6) by equating coefficients in front of the differentials of the independent variables $(dP^{\alpha},dQ^{\alpha},ds)$ and can be expressed as[1]

$$\begin{aligned}
\left[(x-Q)^{\alpha}\frac{\partial p_0^{\alpha}}{\partial P^{\beta}}\right] &= \left[(p-p_0)^{\alpha}\frac{\partial (x-Q)^{\alpha}}{\partial P^{\beta}}\right] - \frac{\partial U}{\partial P^{\beta}},\\
(p-P)^{\beta} &= -\left[(p-p_0)^{\alpha}\frac{\partial (x-Q)^{\alpha}}{\partial Q^{\beta}}\right] + \frac{\partial U}{\partial Q^{\beta}} + \left[(x-Q)^{\alpha}\frac{\partial p_0^{\alpha}}{\partial Q^{\beta}}\right],\\
K-H &= -\left[(p-p_0)^{\alpha}\frac{\partial (x-Q)^{\alpha}}{\partial s}\right] + \frac{\partial U}{\partial s} + \left[(x-Q)^{\alpha}\frac{\partial p_0^{\alpha}}{\partial s}\right].
\end{aligned} \tag{7}$$

The distribution function in the new coordinates $F\ (Q,\ P,\ s)$ is related to the distribution function in the old coordinates $f\,(x,\,p,\,s)$ by

$$F(Q,P,s)\ DQDP = f(x,p,s)\ DxDp. \tag{8}$$

Equation (8) expresses particle conservation in the phase-space volume $DxDp$ under the transformation given by Eq. (5). For a canonical transformation, the phase-space volume is conserved according to $DxDp = DQDP$, and therefore $F\ (Q,\ P,\ s) = f[x\ (Q,\ P,\ s),\ p\ (Q,\ P,\ s),\ s]$. The distribution function in the new variables satisfies the Vlasov equation

$$\frac{dF}{ds} = 0. \tag{9}$$

The assumed slow time dependence of the transformed Hamiltonian $K(Q,P,\epsilon s,\epsilon^2 s,\ldots)$ arises from the slow time dependence of the distribution function $F(Q,P,\epsilon s,\epsilon^2 s,\ldots)$ in the new variables.

Following the procedure outlined in Ref. 1, we express

$$p = p_0(Q,P,s) + \sum_{n=1}^{\infty}\epsilon^n p_n, \quad x = Q + \sum_{n=1}^{\infty}\epsilon^n x_n,$$
$$U = U_0(Q,P,s) + \sum_{n=1}^{\infty}\epsilon^n U_n, \quad K = K_0(Q,P,s) + \sum_{n=1}^{\infty}\epsilon^n K_n, \tag{10}$$

where $p_n\ (Q,\ P,\ s)$, $x_n\ (Q,\ P,\ s)$, $U_n\ (Q,\ P,\ s)$, and $K_n\ (Q,\ P,\ s)$ $(n = 0,\ 1,\ 2,\ \ldots)$ are functions to be determined from the iterative procedure. Using Eq. (10), we expand the Hamiltonian H in Eq. (3) according to

$$H(x,p,s) = \sum_{n=0}^{\infty}\epsilon^n H_n(Q,P,s) \tag{11}$$

To take into account multiple time-scale dependence of all functions in Eqs. (10) and (11), i.e., $g(s,\epsilon s,\epsilon^2 s,\ldots)$, we introduce the time hierarchy, $\epsilon^n s = s_n$, $n = 0,\ 1,\ 2,\ \ldots$, and expand the time derivatives as

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial s_0} + \sum_{n=1}^{\infty}\epsilon^n\frac{\partial}{\partial s_n}. \tag{12}$$

Substituting the expansions [Eqs. (10)–(12)] into Eq. (7), we obtain

$$H_0(Q,P,s) = \frac{\kappa(s)[\eta^\alpha Q^\alpha Q^\alpha]}{2},$$
$$p_0^\beta = P^\beta + \frac{\partial U_0}{\partial Q^\beta},$$
$$\frac{\partial U_0}{\partial P^\beta} = 0, \tag{13}$$

$$K_0 - H_0 = \frac{\partial U_0}{\partial s}. \tag{14}$$

From Eq. (14), we obtain

$$K_0 = < H_0 > = < \kappa > \frac{[\eta^\alpha Q^\alpha Q^\alpha]}{2} = 0, \tag{15}$$

$$U_0 = -\kappa^{(1)}\frac{[\eta^\alpha Q^\alpha Q^\alpha]}{2}, \tag{16}$$

and

$$p_0^\beta = P^\beta - \kappa^{(1)}\eta^\beta Q^\beta, \tag{17}$$

where $< A > \equiv (1/S)\int_{s_0}^{s_0+S} d\bar{s}_0 A(\bar{s}_0)$ and $\ll A \gg \equiv A - < A >$. Here, we also introduce the notation $A^{(0)} \equiv \ll A \gg$ and

$$A^{(n)} \equiv \left\langle\!\!\left\langle \int ds_0 A^{(n-1)} \right\rangle\!\!\right\rangle \tag{18}$$

for $n \geq 1$. Substituting the expansions [Eqs. (10) and (11)] into Eqs. (7), we obtain

$$K_n = Z_n + \frac{\partial U_n}{\partial s_0} - \sum_{l=1}^{n-1}\left[p_{n-l}^\alpha\frac{\partial x_l^\alpha}{\partial s_0}\right]$$
$$+ \sum_{m=1}^{n-1}\left\{\frac{\partial U_{n-m}}{\partial s_m} - \sum_{l=1}^{n-m-1}\left[p_{n-l-m}^\alpha\frac{\partial x_l^\alpha}{\partial s_m}\right]\right\}, \tag{19}$$

$$x_n^\beta = -\frac{\partial U_n}{\partial P^\beta} + \sum_{l=1}^{n-1}\left[p_{n-l}^\alpha\frac{\partial x_l^\alpha}{\partial P^\beta}\right];$$
$$p_n^\beta = \frac{\partial U_n}{\partial Q^\beta} - \sum_{l=1}^{n-1}\left[p_{n-l}^\alpha\frac{\partial x_l^\alpha}{\partial Q^\beta}\right] - \kappa^{(1)}\eta^\beta x_n^\beta, \tag{20}$$

where $n = 1,\ 2,\ \ldots$ and $Z_n \equiv H_n - \kappa[\eta^\alpha x_n^\alpha Q^\alpha]$. For the Hamiltonian function $H\ (x,\ p,\ s)$ [Eq. (3)], the functions Z_n depend only on p_k and x_k, with $k < n$. The solution of Eqs. (19) and (20) can be found iteratively subject to the requirement that $\ll K_n \gg = 0$ and $< x_n > = 0$.

Equations (19) and (20) differ from the equations analyzed in Ref. 1 by the presence of terms inside the curly bracket in Eq. (19). Here, we study the corrections that those terms may contribute to the transformed Hamiltonian K up to the third order in ϵ (to terms K_1, K_2, K_3) and the canonical transformation equations up to the second order in ϵ (to terms $x_1^\alpha, x_2^\alpha, p_1^\alpha, p_2^\alpha$). Because $K_0 = 0$, the average Hamiltonian K has the form $K = \epsilon(K_1 + \epsilon K_2 + \epsilon^2 K_3 + \cdots)$. The ϵ in front of the bracket renormalizes the time scale, so that the average dynamics occurs on the slow time-scale $Q = Q(\epsilon s)$ and $P = P(\epsilon s)$. Therefore, to determine the trajectories $x(s)$ and $p(s)$ valid to second order in ϵ, we need to determine the average Hamiltonian K valid up to the third order in ϵ.

The additional terms, which we denote here as M_1, M_2, and M_3 are

$$M_1 = 0,$$
$$M_2 = \frac{\partial U_1}{\partial s_1},$$
$$M_3 = \frac{\partial U_2}{\partial s_1} + \frac{\partial U_1}{\partial s_2} - \left[p_1^\alpha\frac{\partial x_1^\alpha}{\partial s_1}\right] = \frac{\partial U_2}{\partial s_1} + \frac{\partial U_1}{\partial s_2}$$
$$+ \left[p_1^\alpha\frac{\partial}{\partial P^\alpha}\right]\left(\frac{\partial U_1}{\partial s_1}\right). \tag{21}$$

It follows from Eqs. (19) and (20) that

$$U_1 = -Z_1^{(1)},$$
$$x_1^\beta = -\frac{\partial U_1}{\partial P^\beta} = \frac{\partial Z_1^{(1)}}{\partial P^\beta},$$
$$p_1^\beta = \frac{\partial U_1}{\partial Q^\beta} - \kappa^{(1)}\eta^\beta x_1^\beta = -\frac{\partial Z_1^{(1)}}{\partial Q^\beta} - \kappa^{(1)}\eta^\beta \frac{\partial Z_1^{(1)}}{\partial P^\beta}. \quad (22)$$

Since ${Z_1}^{(1)}$ has no contribution from the self-field term in the Hamiltonian H in Eq. (3), the terms $x_1^\beta, p_1^\beta, U_1$ are independent of the slowly changing distribution function $F\,(P, Q, s_1, s_2, \ldots)$. Therefore, $\partial U_1/\partial s_n = 0$, for $n = 1, 2, \ldots$, and the correction terms reduce to $M_1 = 0$, $M_2 = 0$. Therefore, there is no correction to the expressions for the average Hamiltonian and canonical transformation up to second order in ϵ, and the correction to the average Hamiltonian in third order is given by $< M_3 >= \partial < U_2 > /\partial s_1$.

From Eq. (19), we obtain

$$U_2 = \left[p_1^\alpha \frac{\partial x_1^\alpha}{\partial s_0}\right]^{(1)} - Z_2^{(1)} + \overline{U}_2(Q, P, s_1), \quad (23)$$

and therefore

$$< M_3 >= \frac{\partial < U_2 >}{\partial s_1} = \frac{\partial \overline{U}_2(Q, P, s_1)}{\partial s_1}, \quad (24)$$

where $\overline{U}_2$ is determined by Eq. (20) and $< x_2 >= 0$ or

$$\frac{\partial \overline{U}_2(Q, P, s_1)}{\partial P^\beta} = \left\langle \left[p_1^\alpha \frac{\partial x_1^\alpha}{\partial P^\beta}\right]\right\rangle. \quad (25)$$

Since, x_1^β, p_1^β are independent of s_1, it follows that $\partial \overline{U}_2/\partial s_1 = 0$, and therefore $< M_3 >= 0$.

Since the correction terms are identically zero, we can reuse the formulas obtained in Ref. 1 for the canonical transformation correct to second order in ϵ and the average Hamiltonian correct to third order in ϵ by replacing the time-independent average Hamiltonian with a Hamiltonian, which is a slowly varying function of time through its dependence on the slowly varying distribution function $F(P, Q, \epsilon s)$. This gives

$$x^\alpha = Q^\alpha - \epsilon\kappa^{(2)}\eta^\alpha Q^\alpha + \epsilon^2\left\{2\kappa^{(3)}\eta^\alpha P^\alpha + \left(\kappa\kappa^{(2)}\right)^{(2)} Q^\alpha\right\}, \quad (26)$$

and

$$\begin{aligned} p^\alpha =& \left\{P^\alpha - \kappa^{(1)}\eta^\alpha Q^\alpha\right\} \\ &+ \epsilon\left\{\kappa^{(2)}\eta^\alpha P^\alpha + \left(\kappa\kappa^{(2)}\right)^{(1)} Q^\alpha\right\} + \epsilon^2\Bigg\{\Bigg(3 < (\kappa^{(2)})^2 > \\ &- 2(\kappa\kappa^{(3)})^{(1)} - (\kappa\kappa^{(2)})^{(2)}\Bigg) P^\alpha \\ &+ \left(\kappa^{(3)} < (\kappa^{(1)})^2 > - (\kappa(\kappa\kappa^{(2)})^{(2)})^{(1)}\right)\eta^\alpha Q^\alpha\Bigg\} \\ &+ \epsilon \frac{\partial}{\partial Q^\alpha}\Psi(Q, s)^{(1)}, \end{aligned} \quad (27)$$

with inverse transformation given by

$$\begin{aligned} Q^\alpha =& x^\alpha + \epsilon\kappa^{(2)}\eta^\alpha x^\alpha + \epsilon^2\Big\{-2\kappa^{(3)}\eta^\alpha p^\alpha + (3 < \left(\kappa^{(2)}\right)^2 > \\ &-2\left(\kappa\kappa^{(3)}\right)^{(1)} - \left(\kappa\kappa^{(2)}\right)^{(2)})x^\alpha\Big\}, \end{aligned} \quad (28)$$

and

$$\begin{aligned} P^\alpha =& \left\{p^\alpha + \kappa^{(1)}\eta^\alpha x^\alpha\right\} - \epsilon\left\{\kappa^{(2)}\eta^\alpha p^\alpha + \left(\kappa\kappa^{(2)}\right)^{(1)} x^\alpha\right\} \\ &+ \epsilon^2\Big\{\left(\kappa\kappa^{(2)}\right)^{(2)} p^\alpha - \Big(\kappa^{(3)} < \left(\kappa^{(1)}\right)^2 > \\ &- \left(\kappa\left(\kappa\kappa^{(2)}\right)^{(2)}\right)^{(1)}\Big)\eta^\alpha x^\alpha\Big\} - \epsilon\frac{\partial}{\partial x^\alpha}\Psi(x, s)^{(1)}. \end{aligned} \quad (29)$$

The slow time-dependent Hamiltonian is then determined to be (correct to the third order in ϵ)

$$\begin{aligned} K = \epsilon\Bigg\{ & \frac{[P^\alpha P^\alpha]}{2}\left(1 + 3\epsilon^2 < \left(\kappa^{(2)}\right)^2 >\right) \\ &+ \frac{[Q^\alpha Q^\alpha]}{2}\Big(< \left(\kappa^{(1)}\right)^2 > + \epsilon < \kappa\left(\kappa^{(2)}\right)^2 > \\ &+ \epsilon^2 < \left(\left(\kappa\kappa^{(2)}\right)^{(1)}\right)^2 >\Big) + < \Psi(Q, s) > \Bigg\}. \end{aligned} \quad (30)$$

For a periodic lattice with *odd* half-lattice-period symmetry, $\kappa(s) = -\kappa(s + S/2)$, the term $< \kappa(\kappa^{(2)})^2 >$ occurring in Eq. (30) is identically zero, i.e., $< \kappa(\kappa^{(2)})^2 > \equiv 0$. In Eqs. (26)–(30), the function $\Psi(Q, s)$ is defined correct to second order in ϵ as

$$\begin{aligned} \Psi(Q, s) = \int & D\bar{Q} L(Q_\alpha(1 + \epsilon\eta^\alpha\kappa^{(2)}), \\ & \bar{Q}_\alpha(1 + \epsilon\eta^\alpha\kappa^{(2)})) n(\bar{Q}, \epsilon s), \end{aligned} \quad (31)$$

where $n(Q, \epsilon s) = \int DPF(Q, P, \epsilon s)$.

In Ref. 1, we obtained the expressions for $< \Psi(Q, s) >$ correct to the second order in ϵ,

$$\begin{aligned} < \Psi(Q, s) > = \phi_0 + \epsilon^2 < (\kappa^{(2)})^2 > \Bigg(& \phi_1 - \left[\eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha}\right]\phi_2 \\ &+ \frac{1}{2}\left[\eta^\alpha\eta^\beta Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta}\right]\phi_0\Bigg), \end{aligned} \quad (32)$$

and $\Psi(Q, s)^{(1)}$ correct to the first order in ϵ,

$$\Psi(Q, s)^{(1)} = \epsilon\kappa^{(3)}\left(\phi_2 + \left[\eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha}\right]\phi_0\right), \quad (33)$$

where the functions $\phi_0(Q, \epsilon s)$, $\phi_1(Q, \epsilon s)$, and $\phi_2(Q, \epsilon s)$ satisfy the Poisson-like equations

$$\begin{aligned} \nabla_\perp^2 \phi_0 &= -s_b n(Q, \epsilon s), \\ \nabla_\perp^2 \phi_1 &= -s_b\Bigg(1 + \left[Q^\alpha \frac{\partial}{\partial Q^\alpha}\right] \\ &\qquad + \frac{1}{2}\left[\eta^\alpha\eta^\beta Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta}\right]\Bigg) n(Q, \epsilon s), \\ \nabla_\perp^2 \phi_2 &= -s_b\left[\eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha}\right] n(Q, \epsilon s), \end{aligned} \quad (34)$$

where $\nabla_\perp^2 \equiv (\partial/\partial Q^1)(\partial/\partial Q^1) + (\partial/\partial Q^2)(\partial/\partial Q^2)$ and

$$\nabla_\perp^2 L(Q,\bar{Q}) = -s_b \delta(Q-\bar{Q}). \tag{35}$$

To solve Eqs. (34), one needs to specify some boundary surface in the coordinate space (Q^1, Q^2) and certain boundary conditions on this boundary. It is convenient to designate this boundary surface to be a surface in the coordinate space (Q^1, Q^2), where the function $L(Q,\bar{Q})$ satisfies the same boundary conditions as the function $L(x,\bar{x})$ in the coordinate space (x^1, x^2). In that case, the boundary conditions for $\phi_0(Q)$, $\phi_1(Q)$ and $\phi_2(Q)$ in Eq. (34) in the coordinate space (Q^1, Q^2) are the same as the boundary conditions for the Green's function $L(Q,\bar{Q})$. Note that this boundary surface in the coordinate space (Q^1, Q^2) becomes a surface that oscillates around the boundary surface in the coordinate space (x^1, x^2). Because the two surfaces differ, the average potential in the coordinate space (Q^1, Q^2) does not satisfy the same boundary conditions as the un-averaged potential in the coordinate space (x^1, x^2), as can be seen from Eq. (32).

Note from Eqs. (26)–(30) and the definitions in Eq. (18), that the actual expansion parameter in Eqs. (26)–(30) is not ϵ, but rather $\bar{\epsilon} \equiv \epsilon[<(\kappa^{(1)})^2>]^{1/2} \sim \sigma_v/(2\pi)$. For a lattice with small filling factor $\delta \sim T/S \ll 1$, when the focusing elements occupy a distance $2T$ which is a small portion of the lattice period S, the correction $[<(\kappa^{(1)})^2>]^{1/2} \sim \delta$ can be quite important. For such lattices, the theory presented in this paper still applies even if $\epsilon > 1$, provided the condition $\epsilon\delta \ll 1$ still holds. It can be easily shown that for intense beams with normalized intensity $s_b \lesssim 1$, the self-field part of the average Hamiltonian K is of the same order as the external focusing part, which is in turn of the same order as the kinetic part

$$<\Psi(Q,s)> \sim \frac{[Q^\alpha Q^\alpha]}{2} < (\kappa^{(1)})^2 > \sim \frac{[P^\alpha P^\alpha]}{2}, \tag{36}$$

then the self-field terms in the expressions for the canonical transformation in Eqs. (26)–(29) have an order which is consistent with expansion in the small parameter $\sigma_v/(2\pi) \sim \epsilon\delta$.

IV. MATCHED THERMAL QUASI-EQUILIBRIUM SOLUTION

Equations (32) and (34) determine the average self-field potential for arbitrary distribution function which is a slow function of time when expressed in the slow phase-space coordinates (Q, P). In Ref. 1, we also derived the equations for the average self-field potential for the equilibrium time-independent distribution function which is a function of the average Hamiltonian only $F = G[K(Q,P)]$. This case corresponds to a beam which is matched to the lattice, with all beam quantities changing periodically with time $f(x, p, s+S) = f(x, p, s)$. Here, we study in more details the particular case of a thermal equilibrium distribution, i.e.,

$$F(Q,P) = \frac{\bar{n}[1+3\epsilon^2 < (\kappa^{(2)})^2 >]}{2\pi T} \exp\left(-\frac{K(Q,P)/\epsilon}{T}\right) \tag{37}$$

where T is the normalized temperature and $\bar{n}$ is the normalized on-axis density. It is of particular interest to consider a perfectly conducting cylindrical boundary which is located at a distance R from the beam center. In this case, $L(x,\bar{x}) = 0$ on the surface $(x^1)^2 + (x^2)^2 = R^2$. As explained above, the boundary surface in coordinate space (Q^1, Q^2) is given by the condition $L(Q,\bar{Q}) = 0$ and therefore is given by $(Q^1)^2 + (Q^2)^2 = R^2$. On this surface, $\phi_0 = \phi_1 = \phi_2 = 0$. Note, that this surface is not the conductor boundary, which is given by $(x^1)^2 + (x^2)^2 = R^2$.

It is convenient to introduce the cylindrical coordinates (r,θ), where $Q^1 = r\cos(\theta)$ and $Q^2 = r\sin(\theta)$. It has been shown in Ref. 1 that the averaged potential can be expressed as $<\Psi> = \Phi_0(r) + \epsilon^2 < (\kappa^{(2)})^2 > [p(r) + \cos(4\theta)q(r)]$, where

$$\begin{aligned}
&\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + s_b n_0'(r)\right)p(r) = -s_b\frac{8}{R^4}\int_0^R dr r^3 n_0(r),\\
&\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{16}{r^2} + s_b n_0'(r)\right)q(r)\\
&\quad = -2s_b\left(n_0 + \frac{4}{r^2}\int_0^r d\bar{r}\bar{r}n_0(\bar{r}) - \frac{12}{r^4}\int_0^r d\bar{r}\bar{r}^3 n_0(\bar{r})\right),
\end{aligned} \tag{38}$$

with boundary conditions,

$$\begin{aligned}
p(R) &= -\frac{2s_b}{R^2}\int_0^R dr r^3 n_0(r),\\
q(R) &= -s_b\left(\frac{2}{R^2}\int_0^R d\bar{r}\bar{r}^3 n_0(\bar{r}) - \frac{1}{2}\int_0^R d\bar{r}\bar{r}n_0(\bar{r})\right),
\end{aligned} \tag{39}$$

and for a thermal equilibrium distribution $n_0(r)$ and $\Phi_0(r)$ are determined from

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Phi_0(r) = -s_b n_0(r),\\
&n_0(r) = \bar{n}\exp\left(-\frac{k^2r^2/2 + \Phi_0(r)}{T}\right),
\end{aligned} \tag{40}$$

where $k^2 = < (\kappa^{(1)})^2 > + \epsilon^2 < ((\kappa\kappa^{(2)})^{(1)})^2 >$. Note from Eqs. (38) and (39) that $p(R) \neq 0$ and $q(R) \neq 0$, and therefore $<\Psi> \neq 0$ on the boundary $r = R$. This is because for a quadrupole channel, the boundary $r = R$ is not a real conductor surface, but a surface that oscillates around the conductor surface with an amplitude $\delta R/R \sim \epsilon[<(\kappa^{(2)})^2>]^{1/2}$.

In Eq. (38), $n_0'(r) \equiv dn_0/d\Phi_0 = -n_0(r)/T$. Here, we specialize to the case where the boundary is removed far away from the beam, so that $R \to \infty$ in Eqs. (38) and (39). In this case, $p(r) \equiv 0$. By introducing re-normalized distance $\rho^2 = r^2 s_b\bar{n}/T$ and re-normalized potentials $\bar{\phi}_0 = \Phi_0/T$, $\bar{q} = q/T$, and beam intensity parameter $\bar{s} = s_b\bar{n}/(2k^2)$, Eqs. (38) and (39) can be rewritten as

$$\begin{aligned}
&\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{16}{\rho^2} - \bar{n}_0(\rho)\right)\bar{q}(r)\\
&\quad = -2\left(\bar{n}_0 + \frac{4}{\rho^2}\int_0^\rho d\bar{\rho}\bar{\rho}\bar{n}_0(\bar{\rho}) - \frac{12}{\rho^4}\int_0^\rho d\bar{\rho}\bar{\rho}^3\bar{n}_0(\bar{\rho})\right),
\end{aligned} \tag{41}$$

with boundary conditions

$$\bar{q}(\infty) = \frac{1}{2}\int_0^\infty d\bar{\rho}\bar{\rho}\bar{n}_0(\bar{\rho}), \quad \bar{q}(0) = 0, \tag{42}$$

where the normalized density profile $\bar{n}_0 = n_0/\bar{n}$ satisfies the nonlinear equation

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}\bar{n}_0 + \bar{n}_0\left(\frac{1}{\bar{s}} - \bar{n}_0\right) - \frac{(\bar{n}_0')^2}{\bar{n}_0} = 0. \tag{43}$$

Note that the solutions of Eqs. (41)–(43) depend only on the dimensionless beam intensity parameter $\bar{s}$. Making use of the above definitions, the normalized equilibrium density profile, accurate to second order in ϵ, can be expressed as

$$\bar{n}(\rho,\theta) = \bar{n}_0(\rho)[1 - \epsilon^2 < (\kappa^{(2)})^2 > \cos(4\theta)\bar{q}(\rho)]. \tag{44}$$

Next, we introduce the RMS radius $\rho_{rms}(\theta)$, defined separately for each azimuthal angle θ as

$$\rho^2_{rms}(\theta) \equiv \frac{\int_0^\infty d\rho\rho^3\bar{n}(\rho,\theta)}{\int_0^\infty d\rho\rho\bar{n}(\rho,\theta)}. \tag{45}$$

Using Eq. (44), the relative change in the RMS beam radius as function of angle θ can be expressed as

$$\delta\rho_{rms}(\theta)/\rho^0_{rms} = -\epsilon^2 < (\kappa^{(2)})^2 > \cos(4\theta)A(\bar{s}), \tag{46}$$

where the function $A(\bar{s})$ is given by

$$A(\bar{s}) = \frac{1}{2}\left(\frac{\int_0^\infty d\rho\rho^3\bar{n}_0(\rho)\bar{q}(\rho)}{\int_0^\infty d\rho\rho^3\bar{n}_0(\rho)} - \frac{\int_0^\infty d\rho\rho\bar{n}_0(\rho)\bar{q}(\rho)}{\int_0^\infty d\rho\rho\bar{n}_0(\rho)}\right). \tag{47}$$

Equation (43) for the density profile $\bar{n}_0$ has been analyzed in Ref. 21. There, it was shown that the solution satisfying boundary conditions $\bar{n}_0(0) = 1$ and $\bar{n}_0(\infty) = 0$ exists only for values of beam intensity parameter $\bar{s} < 1$. For low-intensity beams, $\bar{s} \ll 1$, the solution can be approximated by

$$\bar{n}_0(\rho) \approx \left(1 + \frac{\rho^2}{4\bar{s}}\right)^{\bar{s}} \exp\left(-\frac{\rho^2}{4\bar{s}}\right), \tag{48}$$

and for high-intensity beams with $\Delta \equiv 1/\bar{s} - 1 \ll 1$, the solution can be approximated by

$$\bar{n}_0(\rho) \approx \frac{\left[1 + \frac{1}{2}\Delta + \frac{1}{24}\Delta^2\right]^2}{\left[1 + \frac{1}{2}\Delta I_0(\rho) + \frac{1}{24}[\Delta I_0(\rho)]^2\right]^2}, \tag{49}$$

where $I_0(\rho)$ is the modified Bessel function of zero order. The beam density profiles $\bar{n}_0(\rho)$ obtained by numerical integration of Eq. (43) are plotted in Fig. 1 for several values of the beam intensity parameter $\bar{s}$ and compared with the approximate expressions in Eqs. (48) and (49). These density profiles have been used to numerically integrate Eq. (41), and the results are illustrated in Fig. 2 where the function $\bar{n}_0(\rho)\bar{q}(\rho)$, which is the radial profile of the perturbation in the density profile [Eq. (44)], is plotted for different values of the intensity parameter $\bar{s} = 0.3; 0.6; 0.9; 0.99; 0.999$.

The function $A(\bar{s})$, which determines the intensity dependence of the angle-dependent corrections [see Eq. (46)], is plotted in Fig. 3 together with the estimate that is obtained from the definition in Eq. (47). Here, for low-intensity beams with $\bar{s} \ll 1$, it follows from Eqs. (41), (42), and (48) that the function $\bar{q}$ behaves as $\bar{q} = \bar{s}f[\rho^2/(4\bar{s})]$, and therefore it follows from Eq. (47) that $A(\bar{s}) \approx \alpha\bar{s}$. The direct calculations in this limit shows that $\alpha \approx 0.045$. For intense beams with $\Delta \equiv 1/\bar{s} - 1 \ll 1$, the density profile is almost flat $\bar{n}_0 \approx 1$ for all values of $\rho \lesssim \rho_{1/2}$ and reduces to zero over the distance $\delta\rho \sim 1 \ll \rho_{1/2}$. Here, $\rho_{1/2}$ is located at the beam edge where $\bar{n}_0(\rho_{1/2}) = 1/2$. For all values of ρ where the profile is flat, the right-hand side of Eq. (41) is zero, and therefore $\bar{q} \approx 0$. In the region away from the beam edge where $\bar{n}_0 \approx 0$, the function $\bar{q}(\rho)$ behaves as

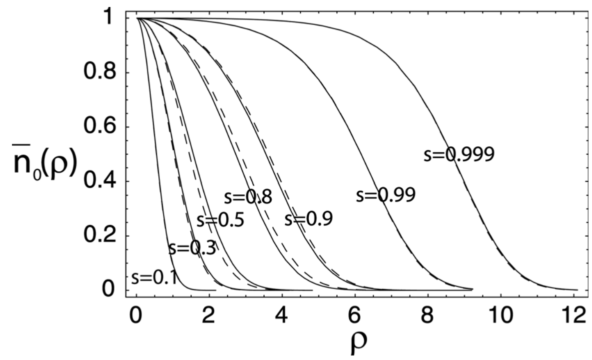

FIG. 1. Plots of density profiles $\bar{n}_0(\rho)$ for different values of the intensity parameter $\bar{s} = 0.1; 0.3; 0.5; 0.8; 0.9; 0.99; 0.999$. The solid lines correspond to the numerical solutions of Eq. (43). The dashed lines correspond to the approximate expressions in Eq. (48) for $\bar{s} = 0.1; 0.3; 0.5$ and in Eq. (49) for $\bar{s} = 0.8; 0.9; 0.99; 0.999$.

$$\bar{q}(\rho) = \bar{q}(\infty)\left[1 - 2\left(\frac{\rho^0_{rms}}{\rho}\right)^2\right]^2. \tag{50}$$

For an almost flat density profile, $2(\rho^0_{rms})^2 \approx \rho^2_{1/2}$ and $\bar{q}(\infty) \approx \rho^2_{1/2}/4$. Using the estimate $\bar{q}(\rho_{1/2} + \delta\rho) \approx \bar{q}(\infty)(2\delta\rho/\rho_{1/2})^2 \approx (\delta\rho)^2 \sim 1$, we obtain from Eq. (47) that

$$A(\bar{s}) \approx \frac{\beta}{\rho_{1/2}(\bar{s})}. \tag{51}$$

The $\bar{s} = 1/(1 + \Delta)$ dependence of edge radius is determined from Eq. (49) and $\bar{n}_0(\rho_{1/2}) = 1/2$ and can be approximated by[21]

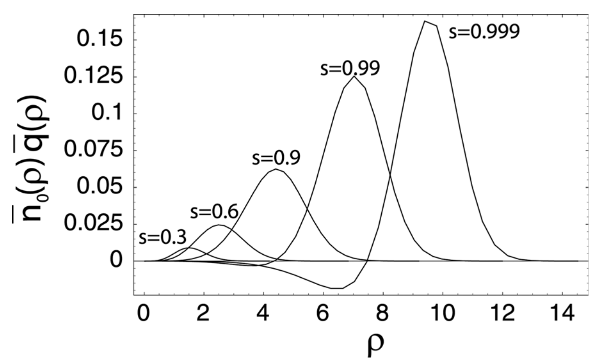

FIG. 2. Plots of $\bar{q}(\rho)\bar{n}_0(\rho)$ for different values of the intensity parameter $\bar{s} = 0.3; 0.6; 0.9; 0.99; 0.999$ obtained from numerical solutions to Eqs. (41) and (42).

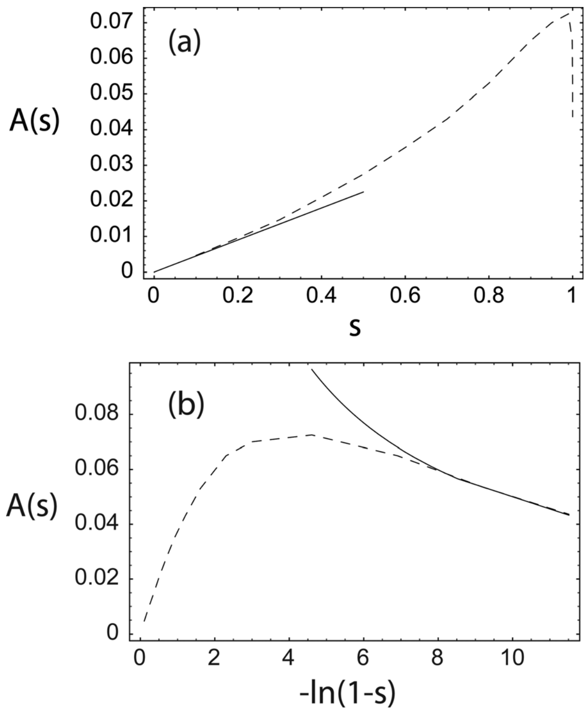

FIG. 3. Plots of the function $A(\bar{s})$ in Eq. (47): (a) linear s-scale and (b) logarithmic s-scale. The dashed lines are obtained numerically, and the solid lines are theoretical estimates.

$$\rho_{1/2}(\bar{s}) \simeq \ln\left[\frac{C}{\Delta}\sqrt{2\pi\frac{C}{\Delta}}\right], \tag{52}$$

where $C \simeq 0.77799$, with the proportionality coefficient β found numerically to be $\beta \approx 0.57$.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

In conclusion, in this paper we have studied the nonlinear transverse dynamics of an intense charged particle beam propagating inside a periodic FODO lattice with period S and characteristic focusing lattice strength κ_0. In our recent work,[1] the perturbative canonical Hamiltonian transformation method was developed to study the quasi-equilibrium solutions of the nonlinear Vlasov-Poisson's system of equation describing a matched intense charged particle beam, where the self-field of the beam was allowed to be of the same order as the average focusing potential of the FODO lattice. The method consists of finding a canonical transformation [Eq. (5)] which is used to transform away the fast particle oscillations with lattice period S. The dynamics of the new phase-space coordinates (Q, P) are described by the transformed time-independent Hamiltonian $K\ (Q, P)$. The time-independent solution of the Vlasov equation in the new coordinates $F\ (Q, P)$ corresponds to the periodic matched quasi-equilibrium solution in the original coordinates $f\,(x, p, s+S) = f\,(x, p, s)$, where $f\,(x, p, s)$ is the beam distribution function. The canonical transformation, accurate to the second order in small parameter $\bar{\epsilon} \sim \sigma_v/2\pi$, where σ_v is the vacuum phase advance,[6] has been determined for arbitrary periodic FODO lattice and equations describing the equilibrium self-field potential accurate to second order in the small parameter $\bar{\epsilon}$ has been derived for arbitrary boundary conditions. As was shown in Ref. 1, the derived formulas allow us to extend the average formulaic results to larger values of vacuum phase advances approaching $\sigma_v \sim 90°$, with accuracies to within several percent.

In this paper, we have used a multi-scale analysis to generalize our previous treatment of the quasi-equilibrium beam to the case where the transformed Hamiltonian is allowed to change slowly with time, $K\,(Q, P, \bar{\epsilon}s, \bar{\epsilon}^2 s, \ldots)$, through the possible slow-time dependence of the transformed distribution function $F\,(Q, P, \bar{\epsilon}s, \bar{\epsilon}^2 s, \ldots)$. We have shown that the change, compared to the quasi-equilibrium case, in the expressions for the canonical transformations, correct to second order in $\bar{\epsilon}$, and transformed Hamiltonian correct to third order in $\bar{\epsilon}$, consists of replacing the time-independent distribution function with its instantaneous time-dependent value $F\ (Q,\ P,\ s)$ [see Eqs. (26)–(34)]. The new time-dependent formulation can be used to describe collective beam dynamics which is slow in the transformed coordinates (slow compared to the period of lattice oscillations) and to study the stability properties of the obtained quasi-equilibria. The derived formulae are extensions to the next order in the small parameter $\bar{\epsilon} \sim \sigma_v/2\pi$ of previous well-known results for the average dynamics of an intense charged particle beam in the so called "smooth-focusing" approximation, which is accurate only to first-order in the small parameter $\bar{\epsilon} \sim \sigma_v/2\pi$.[6] The main difference with a "smooth-focusing" approximation that appears in the next order is the contribution that the fast beam particle oscillations make correction to the average beam self-field potential described by Eqs. (32)–(34). Due to the oscillations, the average potential does not satisfy the same boundary conditions as the un-averaged beam self-field potential, and the average potential also has a non-axisymmetrical contribution, even for a beam which is axisymmetric (on average) and is centered inside a circular conducting pipe. For example, for a beam in quasi-equilibrium inside a perfectly conducting pipe with wall radius R, the averaged potential can be expressed in cylindrical coordinates as $<\Psi>= \Phi_0(r) + \epsilon^2 < (\kappa^{(2)})^2 > [p(r) + \cos(4\theta)q(r)]$, where the functions $p(r)$ and $q(r)$ satisfy Eq. (38), with the boundary conditions given by Eq. (39).

As a particular example, in Ref. 1 we studied an intense charged particle beam with Kapchinskij-Vladimirskij distribution[6] inside a perfectly conducting cylindrical pipe with wall radius R. It was shown that the correction to the average self-field potential inside the beam is zero when the boundary is removed to $R \to \infty$. When the boundary location is finite (R is finite), the average motion of the image charges induces non-zero corrections to the average self-field potential inside the beam, which results in the average beam radius acquiring an octupole component which is proportional to $(\sigma_v/2\pi)^2 (a/R)^8$, where a is the average beam radius.

In the present paper, we analyzed the second-order correction to the equilibrium beam density profile for a thermal equilibrium distribution. We have shown that even for the case where the boundary is removed to infinity $R \to \infty$, the density acquires the octupole component in Eq. (44), which in turn produces an octupole variation of the RMS beam radius in Eq. (46). The angle-dependent density correction is localized at the beam edge and reaches its maximum value,

$(\delta\bar{n}/\bar{n})_{max} \sim (\sigma_v/2\pi)^2 B(\bar{s})$ with $[B(\bar{s})]_{max} \sim 0.17$ (see Fig. 2), for intense beams with normalized intensity parameter $\bar{s} \approx 1$.

In both cases, the corrections to the self-field are small, and therefore the "smooth-focusing" approximation for the self-field potential can be a good approximation even for moderate values of the vacuum phase advance. For example, for vacuum phase advance of $\sigma_v = 90°$, the correction to the RMS radius of the beam described by a thermal equilibrium distribution arising from the corrections to the average self-field potential is of order 0.5%. Nonetheless, note that because the average self-field potential acquires an octupole component, the average motion of some beam particles becomes non-integrable and the trajectories become chaotic. This chaotic behavior of some of the beam particles may change the nature of the Landau damping (or growth) of collective excitations supported by the beam. Also, due to the presence of the extra non-axisymmetric terms in the equations for the self-field potential, the stability properties of different beam quasi-equilibria can change significantly.

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