A physical parametrization of coupled transverse dynamics based on generalized Courant–Snyder theory and its applications

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A physical parametrization of coupled transverse dynamics is developed by generalizing the Courant–Snyder (CS) theory for one degree of freedom to the case of coupled transverse dynamics with two degrees of freedom. The four basic components of the original CS theory, i.e., the envelope equation, phase advance, transfer matrix, and CS invariant, all have their counterparts with remarkably similar expressions in the generalized theory. Applications of the new theory are given. It is discovered that the stability of coupled dynamics is completely determined by the generalized phase advance. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3142472]

The transverse dynamics of a charged particle in a linear focusing lattice $\kappa_q(t)$ is described by an oscillator equation with time-dependent spring constant,

$$\ddot{q}+\kappa_q(t)q=0, \tag{1}$$

where q represents one of the transverse coordinates, either x or y. For a quadrupole lattice, $\kappa_x(t)=-\kappa_y(t)$. The Courant–Snyder (CS) theory[1] gives a complete description of the solution to Eq. (1) and serves as the fundamental theory that underlies the design of modern accelerators and storage rings. There are four main components of the CS theory: the envelope equation, the phase advance, the transfer matrix, and the CS invariant. The CS theory can be summarized as follows. Because Eq. (1) is linear, its solution can be expressed as a time-dependent linear map from the initial conditions, i.e., $(q,\dot{q})^\dagger=M(t)(q_0,\dot{q}_0)^\dagger$,

$$M(t)=\begin{pmatrix}\sqrt{\dfrac{\beta}{\beta_0}}[\cos\phi+\alpha_0\sin\phi] & \sqrt{\beta\beta_0}\sin\phi\\ -\dfrac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\phi+\dfrac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\phi & \sqrt{\dfrac{\beta_0}{\beta}}[\cos\phi-\alpha\sin\phi]\end{pmatrix}, \tag{2}$$

where $q_0=q(t=0)$, $\dot{q}_0=\dot{q}(t=0)$, $\beta_0=\beta(t=0)$, and $\alpha_0=\alpha(t=0)$. The superscript "$\dagger$" denotes the transpose operation. The time-dependent functions $\alpha(t)$, $\beta(t)$, and $\phi(t)$ in the transfer matrix $M(t)$ are directly related to the envelope function $w(t)$ by

$$\beta(t)=w^2(t),\quad \alpha(t)=-w\dot{w},\quad \phi(t)=\int_0^t\frac{dt}{\beta(t)}. \tag{3}$$

The envelope function $w(t)$ satisfies the nonlinear envelope equation

$$\ddot{w}+\kappa_q(t)w=w^{-3}. \tag{4}$$

The physical meanings of β^{-1} and ϕ correspond to the phase advance rate and the phase advance, respectively. The transfer matrix $M(t)$ is symplectic and has the following decomposition:[2]

$$M(t)=\begin{pmatrix}w & 0\\ \dot{w} & \dfrac{1}{w}\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}w_0^{-1} & 0\\ -\dot{w}_0 & w_0\end{pmatrix}. \tag{5}$$

The well-known CS invariant[1,3–5] is

$$I=\frac{q^2}{w^2}+(w\dot{q}-\dot{w}q)^2. \tag{6}$$

We emphasize that the CS theory is unique among many possible mathematical schemes to parametrize the symplectic transfer matrix. The parameters of envelope, phase advance, and CS invariant furnished by the CS theory are of vital importance for beam physics. These parameters describe the physical dimensions and the emittance of the beam and set the foundation for many important concepts in beam physics, such as the Kapchinskij–Vladimirskij distribution function for beams with strong space-charge field.[6]

When applying the CS theory to accelerators, the dynamics in the two transverse directions are considered to be decoupled. However, the coupling between the two transverse directions can be of considerable practical importance.[7,8] This effect was actually discussed by Courant and Snyder.[1] The general form of the Hamiltonian for the coupled transverse dynamics is given by

$$H_c=z^\dagger A_c z,\quad z=(x,y,\dot{x},\dot{y})^\dagger, \tag{7}$$

$$A_c = \begin{pmatrix} \kappa & R \\ R^\dagger & \frac{I}{2} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}. \tag{8}$$

Here, the 2×2 matrix $\kappa(t)$ is time dependent and symmetric, R is an arbitrary, time-dependent 2×2 matrix, and I is the 2×2 unit matrix. The transverse dynamics are coupled through the $\kappa_{xy}(t)$ terms and the matrix R. A solenoidal lattice will induce nonvanishing R, and a skew quadrupole field will induce nonvanishing κ_{xy}. For a combined lattice with quadrupole, skew quadrupole, and solenoidal components,

$$\kappa = \begin{pmatrix} \frac{\Omega^2}{2} + \frac{\kappa_q}{2} & \kappa_{\mathrm{sq}} \\ \kappa_{\mathrm{sq}} & \frac{\Omega^2}{2} - \frac{\kappa_q}{2} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -\frac{\Omega}{2} \\ \frac{\Omega}{2} & 0 \end{pmatrix}, \tag{9}$$

where κ_q is the quadrupole focusing coefficient, $\Omega(t)=eB_z(t)/\gamma mc$ is the gyrofrequency associated with the solenoidal lattice, and κ_{sq} is the skew quadrupole coefficient.

The solution of the linear coupled system corresponding to H_c is given by a transfer matrix $M_c(t)$, which is a time-dependent 4×4 symplectic matrix.[1] Because there are ten free parameters for a 4×4 symplectic matrix, many different mathematical parametrization schemes for $M_c(t)$ exist. Teng and co-worker[9–11] first systematically studied the transfer matrix and derived various parametrization schemes,[9] among which the "symplectic rotation form"[10] has been adopted in lattice design and particle tracking codes, such as the MAD code.[12] Other possible parametrizations have also been considered.[13] However, these parametrizations lack connections with the physics of the beam dynamics. They do not provide us with physical insights into the coupled dynamics. For example, these parametrization schemes do not give us effective tools to investigate the stability properties of the coupled dynamics. Moreover, they do not describe the beam envelopes for the coupled transverse dynamics, which are obviously key physical parameters of the beams. Ripken[14,15] developed a method to describe beam envelopes for coupled dynamics without using these parametrization schemes, which attests to the ineffectiveness of the existing parametrization schemes.

In this Letter, we develop a new physical parametrization of the transfer matrix $M_c(t)$ for coupled transverse dynamics by extending the CS theory for one degree of freedom to the case of coupled transverse dynamics described by the Hamiltonian H_c in Eq. (7). We show that the generalized CS theory gives a complete description of the coupled transverse dynamics and has the same structure as the original CS theory for one degree of freedom. The four basic components of the original CS theory that have physical importance, i.e., the envelope equation, phase advance, transfer matrix, and CS invariant, all have their counterparts, with remarkably similar expressions, in the generalized CS theory developed here. The unique feature of the generalized CS theory presented here is the non-Abelian nature of the theory. In the generalized theory, the envelope function w is generalized to an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are not commutative. The generalized theory gives a parametrization of the four-dimensional (4D) symplectic transfer matrix M_c [Eq. (17)] that has the same structure as the parametrization of the two-dimensional (2D) symplectic transfer matrix M [Eq. (5)] in the original CS theory. We will then apply the new parametrization developed in this paper to study the stability of both strongly and weakly coupled dynamics. It is discovered that the stability of the coupled dynamics is completely determined by the generalized phase advance. Two stability criteria are given, which recover the known results about sum and difference resonances in the weakly coupled limit. Due to length restrictions, we present here only the results for the case of the coupled dynamics induced by a skew quadrupole component, i.e., $\kappa_{\mathrm{sq}}\neq 0$, $R=0$, and $\Omega=0$. The more general case with nonvanishing R and Ω, together with specific examples, will be described in detail in future publications.

We use a time-dependent canonical transformation, first proposed by Leach,[16] to develop the generalized CS theory. We consider a linear, time-dependent Hamiltonian system with n degrees of freedom given by $H=z^\dagger A(t)z$ and $z=(x_1,x_2,\ldots,x_n,\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n)^\dagger$. Here, $A(t)$ is a $2n\times 2n$ time-dependent, symmetric matrix. The Hamiltonian in Eq. (7) has this form with $n=2$. We introduce a time-dependent linear canonical transformation $\bar{z}=S(t)z$, such that in the new coordinate $\bar{z}$, the transformed Hamiltonian has the form $\bar{H}=\bar{z}^\dagger\bar{A}(t)\bar{z}$, where $\bar{A}(t)$ is a targeted symmetric matrix. Because the transformation is canonical, the matrix S needs to be symplectic (i.e., $SJS^\dagger=J$) at t. In addition, the time-dependent symplectic matrix S needs to satisfy the differential equation[16,17]

$$\dot{S} = 2(J\bar{A}S - SJA), \tag{10}$$

where J is the $2n\times 2n$ unit symplectic matrix of order $2n$.

We are now ready to develop the generalized CS theory for coupled transverse dynamics described by the Hamiltonian H_c using this technique of time-dependent canonical transformation. Our objective is to solve the coupled system by finding the transfer matrix between the initial conditions $z_0=(x_0,y_0,\dot{x}_0,\dot{y}_0)^\dagger$ and $z=(x,y,\dot{x},\dot{y})^\dagger$ at time t. We accomplish this goal by two time-dependent canonical transformations. The first step is to transform H_c into

$$\bar{H}_c = \bar{z}^\dagger\bar{A}_c\bar{z}, \quad \bar{A}_c = \begin{pmatrix} \frac{\beta^{-1}}{2} & 0 \\ 0 & \frac{\beta^{-1}}{2} \end{pmatrix}, \tag{11}$$

and the second step is to transform $\bar{H}_c$ into $\bar{\bar{H}}_c=0$. Here, β is a time-dependent 2×2 matrix to be determined. As implied by its notation, the matrix of β is the generalized β function for the coupled dynamics. The physics that appears in the first step is the envelope matrix and the non-Abelian matrix envelope equation. The physics that appears in the second step is the phase advance. Let $\bar{z}=Sz$ be the transformation that transforms H_c into $\bar{H}_c$. From Eq. (10), the differential equation for S is

$$\dot{S}=2(J\bar{A}_c S - SJA_c). \tag{12}$$

The solution of Eq. (12) is[17]

$$S=\begin{pmatrix}(w^{-1})^\dagger & 0\\ -\dot{w} & w\end{pmatrix},$$

where $\beta^{-1}=(w^{-1})^\dagger w^{-1}$ and w is the 2×2 envelope matrix satisfying the envelope matrix equation

$$\ddot{w}+w\kappa=(w^{-1})^\dagger w^{-1}(w^{-1})^\dagger. \tag{13}$$

The inverse transformation is

$$z=S^{-1}\bar{z},\quad S^{-1}=\begin{pmatrix}w^\dagger & 0\\ w^{-1}\dot{w}w^\dagger & w^{-1}\end{pmatrix}. \tag{14}$$

The matrix S^{-1} is the non-Abelian generalization of the first matrix in the expression of the transfer matrix M for the original CS theory, i.e., the first term on the right hand side of Eq. (5).

The next step is to transform $\bar{H}_c$ into $\bar{\bar{H}}_c=0$ with $\bar{\bar{A}}_c=0$ by a transformation specified by $\bar{\bar{z}}=P\bar{z}$. Following the same procedure, the differential equation for P is

$$\dot{P}=P\dot{\phi},\quad \dot{\phi}\equiv\begin{pmatrix}0 & -(w^{-1})^\dagger w^{-1}\\ (w^{-1})^\dagger w^{-1} & 0\end{pmatrix}, \tag{15}$$

which admits solution of the form $P=\begin{pmatrix}P_1 & P_2\\ -P_2 & P_1\end{pmatrix}$. From the fact that P belongs to $Sp(4,R)$, we can readily show that $PP^\dagger=I$, and $\mathrm{Det}(P)=1$. Therefore, P corresponds to a rotation in the 4D phase space, $P\in SO(4)$. In this sense, $P^\dagger$ is the 4D non-Abelian generalization of the 2D rotation matrix in the expression of the transfer matrix M for the original CS theory, i.e., the second term on the right hand side of Eq. (5). Because $\dot{\phi}^\dagger=-\dot{\phi}$, it follows that $\dot{\phi}$ belongs to the Lie algebra $so(4)$, i.e., $\dot{\phi}$ is an infinitesimal generator of a 4D rotation. In other words, $\dot{\phi}$ is an "angular velocity" in 4D space, which is equivalent to a phase advance rate in 4D space. The 4D phase advance rate $\dot{\phi}$ is determined from the 2×2 matrix $\beta^{-1}=(w^{-1})^\dagger w^{-1}$, which is remarkably similar to the phase advance rate $\beta^{-1}=1/w^2$ in the original CS theory for one degree of freedom [see Eq. (3)].

Because $\bar{\bar{H}}_c=0$, the dynamics of $\bar{\bar{z}}$ is trivial, i.e., $\bar{\bar{z}}=\bar{\bar{z}}_0$, and we have solved the Hamiltonian system H_c in $\bar{\bar{z}}$. From $\bar{\bar{z}}=PSz$ and $\bar{\bar{z}}=\bar{\bar{z}}_0$, we obtain the linear map between z_0 and z, i.e.,

$$z=S^{-1}P^{-1}\bar{\bar{z}}=S^{-1}P^{-1}\bar{\bar{z}}_0=S^{-1}P^{-1}P_0S_0z_0. \tag{16}$$

Because $P\in SO(4,R)$, without loss of generality we select the initial condition $P_0=P(t=0)=I$ to obtain $z=M_cz_0$,

$$M_c=\begin{pmatrix}w^\dagger & 0\\ w^{-1}\dot{w}w^\dagger & w^{-1}\end{pmatrix}\begin{pmatrix}P_1 & -P_2\\ P_2 & P_1\end{pmatrix}\begin{pmatrix}w_0^{-1\dagger} & 0\\ -\dot{w}_0 & w_0\end{pmatrix}. \tag{17}$$

The transfer matrix M_c in Eq. (17) is the 4D non-Abelian generalization of the transfer matrix in Eq. (5) for one degree of freedom. The similarities between M_c and M is evident from Eqs. (17) and (5). The generalized CS invariant for 4D coupled dynamics corresponding to the original CS invariant is

$$\begin{aligned}I_{CS}&=\bar{\bar{z}}^\dagger\bar{\bar{z}}=z^\dagger S^\dagger P^\dagger PSz=z^\dagger S^\dagger Sz\\ &=q^\dagger w^{-1}w^{-1\dagger}q+(\dot{q}^\dagger w^\dagger-q^\dagger\dot{w}^\dagger)(w\dot{q}-\dot{w}q),\end{aligned} \tag{18}$$

where the phase advance has been removed due to the fact that P is a 4D rotation.

We now show that the generalized CS theory developed for coupled transverse dynamics recovers the original CS theory for dynamics with one degree of freedom as a special case. For the uncoupled transverse dynamics given by H_c with $\kappa_{xy}=0$, κ is diagonal, and the matrix envelope equation Eq. (13) admits solutions with diagonal envelope matrix $w=\begin{pmatrix}w_x & 0\\ 0 & w_y\end{pmatrix}$. Consequently, every matrix in Eq. (13) is diagonal, and the matrix operation is Abelian (commutative). The matrix envelope equation reduces to two decoupled envelope equations of the conventional form for w_x and w_y, i.e., $\ddot{w}_x+w_x\kappa_x=w_x^{-3}$ and $\ddot{w}_y+w_y\kappa_y=w_y^{-3}$. The 2×2 matrix of phase advance rate β^{-1} reduces to a diagonal matrix as well, i.e., $\beta^{-1}=\begin{pmatrix}w_x^{-2} & 0\\ 0 & w_y^{-2}\end{pmatrix}$. The phase advance is $P=\begin{pmatrix}P_1 & P_2\\ -P_2 & P_1\end{pmatrix}$, $P_1=\begin{pmatrix}\cos\phi_x & 0\\ 0 & \cos\phi_y\end{pmatrix}$, and $P_2=-\begin{pmatrix}\sin\phi_x & 0\\ 0 & \sin\phi_y\end{pmatrix}$, where $\dot{\phi}_x=w_x^{-2}$ and $\dot{\phi}_y=w_y^{-2}$ are the phase advances in the x and y directions. The transfer matrix reduces to

$$M=\begin{pmatrix}w_x & 0 & 0 & 0\\ 0 & w_y & 0 & 0\\ \dot{w}_x & 0 & w_x^{-1} & 0\\ 0 & \dot{w}_y & 0 & w_y^{-1}\end{pmatrix}\times\begin{pmatrix}\cos\phi_x & 0 & -\sin\phi_x & 0\\ 0 & \cos\phi_y & 0 & -\sin\phi_y\\ \sin\phi_x & 0 & \cos\phi_x & 0\\ 0 & \sin\phi_y & 0 & \cos\phi_y\end{pmatrix}\times\begin{pmatrix}w_{x0}^{-1} & 0 & 0 & 0\\ 0 & w_{y0}^{-1} & 0 & 0\\ -\dot{w}_{x0} & 0 & w_{x0} & 0\\ 0 & -\dot{w}_{y0} & 0 & w_{y0}\end{pmatrix}. \tag{19}$$

Apparently, the $(x,\dot{x})$ dynamics and the $(y,\dot{y})$ dynamics are decoupled, and the transfer matrices for $(x,\dot{x})$ and $(y,\dot{y})$ extracted from Eq. (19) are identical to that in Eq. (5) for one degree of freedom.

We now apply the theory developed here to study the orbit stability of both strongly coupled and weakly coupled transverse dynamics. Weakly coupled systems naturally arise from misalignment of the magnets. Strongly coupled systems, on the other hand, are designed to intentionally couple the transverse dynamics. For example, in the Neutralizing Drift Compression Experiment II which is being designed and constructed for applications to high energy density physics and heavy ion fusion,[18] strongly coupled systems using either skew quadrupole or solenoidal magnets are currently being investigated as a beam smoothing technique. Another example is the Möbius accelerator,[19] in which the dynamics of two transverse directions is interchanged at a point where $\beta_x=\beta_y$ and $\beta_x'=\beta_y'=0$. The current analysis of the Möbius accelerator assumes that the interchange is realized instantaneously by an infinitely thin lens, either a skew quadrupole and or a solenoidal magnet. This is evidently a rather simplified assumption. The β_x and β_y functions outside the "flipping" point are defined independently by the standard CS theory. To understand the physics at the flipping point, a theory capable of describing how β_x and β_y are coupled in a finite-length magnet is needed. Obviously, the appropriate theory for this purpose needs to furnish a concept of generalized envelopes to realize the function of interchanging β_x and β_y. The theory and the associated generalized envelope matrix w and beta function β developed in this Letter provide exactly such a tool. The most important issue of a strongly coupled system is the stability of the coupled dynamic, an area where our theoretical understanding is very limited, owing to the lack of a physics-based parametrization of the coupled dynamics. Using the theoretical formalism developed in this Letter, we are able to deduce valuable information about the stability of the linear coupled dynamics. From Eq. (17), the one-turn transfer map in a ring is

$$M_{cT}(t) = \begin{pmatrix} w^{\dagger} & 0 \\ w^{-1}\dot{w}w^{\dagger} & w^{-1} \end{pmatrix} P_T \begin{pmatrix} w^{-1\dagger} & 0 \\ -\dot{w} & w \end{pmatrix}, \tag{20}$$

where $P_T(t)$ is the generalized one-turn phase advance matrix. Equation (20) is a statement that $M_{cT}(t)$ is similar to $P_T(t)$, and we conclude that the stability of the coupled dynamics is completely determined by the phase advance matrix $P_T(t)$. This remarkable result significantly simplifies the stability analysis and showcases the physical importance of the phase advance $P_T(t)$. In addition, since $P_T(t)$ is a real 4D rotation, we have the following stability criterion for the coupled linear system: a necessary and sufficient condition for the coupled dynamics to be unstable is that the $P_T(t)$ has an eigenvalue λ with $|\lambda| \neq 1$. A second stability criterion for a strongly coupled system can be discovered by looking at the invariant $z^{\dagger}Uz=\text{const}$,[1] where U is given by

$$U(t) = JM_{cT} - M_{cT}^{\dagger}J = S^{\dagger}P_T^{\dagger}J[P_T^{\dagger} - P_T]P_T S. \tag{21}$$

If $U(t)$ is positive (negative) definite, then the amplitude of z is bounded and the dynamics is stable. Equation (21) indicates that $U(t)$ and $J[P_T^{\dagger}-P_T]$ are congruent. We thus have the following stability criterion determined from the antisymmetric component of the phase advance, i.e., $[P_T^{\dagger}-P_T]$: a sufficient condition for the coupled linear transverse dynamics to be stable is that $J[P_T^{\dagger}-P_T]$ is positive or negative definite. This is because if $J[P_T^{\dagger}-P_T]$ is positive or negative definite, so is $U(t)$. From the fact that $z^{\dagger}Uz=\text{const}$, z cannot grow without bound if $U(t)$ is positive or negative definite.

For the weakly coupled case, we show that the above criteria reduce to the known results of the sum and difference resonances. When the coupling effect is weak, it can be treated as a perturbation to the stable uncoupled dynamics. In order for the perturbed phase advance P_T to have an eigenvalue λ with $|\lambda| \neq 1$ for instability, the unperturbed phase advance P_{T0} must have two pairs of identical eigenvalues. Because

$$P_{T0} = \begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} \cos\phi_x & 0 \\ 0 & \cos\phi_y \end{pmatrix}, \quad P_2 = -\begin{pmatrix} \sin\phi_x & 0 \\ 0 & \sin\phi_y \end{pmatrix}, \tag{22}$$

this means that $\cos\phi_x=\cos\phi_y$, i.e., $\nu_x \pm \nu_y = n$, where ν_x and ν_y are tunes, and n is an integer. This is the familiar sum and difference resonance condition. On the other hand, the sufficient condition for stability determined from $[P_T^{\dagger}-P_T]$ in the weakly coupled case reveals that the difference resonance, or $\nu_x-\nu_y=n$, is stable. This is because a small perturbation due to weak coupling effect does not alter the positive (negative) definite character of $J[P_T^{\dagger}-P_T]$. If the unperturbed $J[P_{T0}^{\dagger}-P_{T0}]$ is positive (negative) definite, so is $J[P_T^{\dagger}-P_T]$. We can easily see that when $\sin\phi_x=\sin\phi_y$, or $\nu_x-\nu_y=n$, $J[P_{T0}^{\dagger}-P_{T0}]$ is positive definite. As a result, the difference resonance is stable. This leads us to the known result that only the sum resonance, i.e., $\nu_x+\nu_y=n$, can be unstable when there is a weak coupling effect. In this sense, the two stability criteria discovered by the theory developed here can be viewed as a generalization of the well-known results about sum and difference resonances for weakly coupled system to a coupled system with arbitrary coupling strength.

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