

# Approximate matched solution for an intense charged particle beam propagating through a periodic focusing quadrupole lattice \*

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## Abstract

The transverse dynamics of an intense charged particle beam propagating through a periodic quadrupole focusing lattice is described by the nonlinear Vlasov-Maxwell system of equations. To find matched-beam quasi-equilibrium distribution functions one needs to determine a dynamical invariant for the beam particles moving in the combined external and self-fields. The standard approach, for sufficiently small phase advance  $\sigma_v/2\pi < 1$ , is to use the smooth-focusing approximation, where the particle dynamics is determined iteratively using the small parameter  $\epsilon = \sigma_v/2\pi < 1$  accurate to order  $\epsilon^3$ . Here  $\sigma_v$  is the vacuum phase advance. In this paper, we present a perturbative Hamiltonian transformation method which is used to transform away the fast particle oscillations, and obtain the average Hamiltonian accurate to order  $\epsilon^5$ . This average Hamiltonian, expressed in the original phase-space variables, is an approximate invariant of the original system, and can be used to determine self-consistent beam equilibria that are matched to the focusing channel.

## INTRODUCTION

There is growing interest in studying detailed properties of intense charged particle beams for particle physics applications, high energy density physics research using intense particle beams, and heavy ion beams for inertial fusion energy and warm dense matter applications, etc. In most of the applications, intense charged particle beams have to be transported over long distances through a focusing channel, which provides transverse particle confinement. In a quadrupole focusing channel, the beam particles experience a transverse linear focusing-defocusing force, which is a periodic function of time in the beam frame. This oscillating force provides the necessary focusing only in an average sense [1]. For intense charged particle beams, this average force must be strong enough to prevent both thermal and space-charge expansion of the beam particles.

Identifying regimes for stable beam propagation has been one of the main challenges of accelerator research. In particular, the development of systematic approaches that are able to treat self-consistently the applied periodic focusing force and the self-field force of the beam particles simultaneously is very important. Several recent investigations [2, 3, 4] have used standard Hamiltonian perturbative methods. With these methods, one searches for the generating function that relates the old set of canonical phase-space variables to the new canonical set. The new canoni-

cal variables are chosen to have a Hamiltonian that is independent of time. In the standard approach, the generating function is a function of the mixed set of variables (old and new). This makes the perturbative analysis quite complicated. In particular, the analysis in Refs. [2, 3] was carried out only to third order in the small parameter  $\epsilon$ . The analysis in Ref. [4] was carried out to 5th order, but the authors appeared to have made an error in the iterative procedure, which invalidates the results. An advantage of the present approach is that instead of using a generating function which is a function of the mixed set of variables, we work with functions that depend on the new non-oscillating set of variables from the beginning. This significantly simplifies the analysis and develops an iterative procedure that makes no reference to the generating function in its final form. The authors in Ref. [3, 4] worked with Poisson's equation, while the author of Ref. [2] worked directly with the Green's function of Poisson's equation. We use the latter approach because it allows for a simpler treatment, and allows us to take into account the boundary conditions quite easily.

The transverse dynamics of the intense charged particle beam can be described by the nonlinear Vlasov-Poisson system of equations for the beam distribution function  $f(x, p, s)$  and the normalized self-field potential  $\Psi(x, t)$ . Here  $s = v_b t$  is the longitudinal coordinate, and  $v_b$  is the directed beam velocity. The function  $f(x, p, s)$  satisfies the nonlinear Vlasov equation [1]

$$\begin{aligned} \frac{df}{ds} &= \frac{\partial f}{\partial s} + \frac{dx^\alpha}{ds} \frac{\partial f}{\partial x^\alpha} + \frac{dp^\alpha}{ds} \frac{\partial f}{\partial p^\alpha} = 0, \quad \text{where} \\ \frac{dx^\alpha}{ds} &= \frac{\partial H}{\partial p^\alpha}, \quad \frac{dp^\alpha}{ds} = -\frac{\partial H}{\partial x^\alpha}, \end{aligned} \quad (1)$$

are the particle equations of motion. The Hamiltonian  $H$  for the particle motion is in a force field which is the sum of a linear externally applied focusing field with the focusing field strength  $\kappa(s)$  changing periodically as function of axial coordinate  $s$  according to  $\kappa(s) = \kappa(s + S)$ , and the self-field potential calculated self-consistently using Poisson's equation.

It is convenient to introduce the re-normalized variables  $\bar{x} = x/a$ ,  $\bar{s} = s/S$ ,  $\bar{\kappa}(s) = \kappa(s)/\kappa_0$ ,  $\bar{p} = p/(a\kappa_0^{1/2})$ ,  $\bar{f} = (f/N)a^4\kappa_0$ , and  $\bar{\Psi} = \Psi/(a^2\kappa_0)$ , where  $S$  is the characteristic period of the applied focusing force,  $a$  is the characteristic transverse beam dimension, and  $\kappa_0$  is the characteristic value of the lattice function  $\kappa(s)$ . Equations (1) maintain the same form in normalized variables, whereas normalized Hamiltonian  $\bar{H}$  takes the form

$$\bar{H}(\bar{x}, \bar{p}, s) = \epsilon \left[ \frac{\bar{p}^\alpha \bar{p}^\alpha}{2} + \bar{\kappa}(\bar{s}) \frac{\bar{x}^\alpha \bar{x}^\alpha}{2} \right]$$

\* Research supported by the U. S. Department of Energy.

$$+ \int L(\bar{x} - \bar{x}') \bar{f}(\bar{x}', \bar{p}', s') D\bar{x}' D\bar{p}' \Big], \quad (2)$$

For simplicity, we suppress variable indices inside functions, and adopt the notation  $x^\alpha x^\alpha \equiv \sum_{\alpha=1}^2 x^\alpha x^\alpha$ , unless mentioned otherwise. Moreover, for multi-dimensional integrals, we adopt the notation  $\int dx_1 dx_2 Z = \int DxZ$ . In Eq. (2),  $\epsilon \equiv S\kappa_0^{1/2}$ , and the Green's function  $L(\bar{x} - \bar{x}')$  satisfies the equation

$$\frac{\partial}{\partial \bar{x}^\alpha} \frac{\partial}{\partial \bar{x}^\alpha} L(\bar{x} - \bar{x}') = -\bar{K} \delta(\bar{x} - \bar{x}'). \quad (3)$$

Here,  $\bar{K} = 2\pi q^2 N / m_b v_b^2 \gamma_b^3 \kappa_0 a^2$  is the normalized beam self-field perviance, which is a measure of the beam space-charge intensity,  $m_b$  and  $q$  are the particle mass and charge, respectively, and  $\gamma_b = (1 - v_b^2/c^2)^{-1/2}$  is the relativistic mass factor. In Eq. (2), the function  $\bar{f}$  is normalized according to  $\int d\bar{x} d\bar{p} \bar{f} = 1$ . In what follows, we assume that all terms inside the square bracket in Eq. (2) are of the same order. In writing the solution to Poisson's equation using the Green's function, which is a function of the difference  $L(|\bar{x} - \bar{x}'|)$ , we have also assumed that the transverse boundaries are infinitely far away. This assumption is not strictly necessary, but is made here for simplicity.

## PERTURBATIVE HAMILTONIAN TRANSFORMATION METHOD

In what follows, we drop the bar notations over the normalized variables. To determine the matched solution of the Vlasov equation (1), we search for a time-dependent canonical transformation of the form [1, 2, 3, 4]  $(x^\alpha, p^\alpha, H, s) \rightarrow (Q^\alpha, P^\alpha, K, s)$

$$x^\alpha = x^\alpha(Q, P, s), \quad p^\alpha = p^\alpha(Q, P, s), \quad (4)$$

with time-independent transformed Hamiltonian  $K(Q, P)$ . For every canonical transformation there is a function  $S$  that satisfies the differential relation

$$p^\alpha dx^\alpha - H ds = dS + P^\alpha dQ^\alpha - K ds. \quad (5)$$

It is convenient to search for a function  $S$  of the form  $S = U + P^\alpha(x - Q)^\alpha$ , where  $U(Q, P, s)$  is a function of the new phase-space variables. The relationships between the old and new set of phase-space coordinates are obtained from Eq. (5), and can be expressed as

$$\begin{aligned} (x - Q)^\beta &= (p - P)^\alpha \frac{\partial (x - Q)^\alpha}{\partial P^\beta} - \frac{\partial U}{\partial P^\beta}, \\ (p - P)^\beta &= -(p - P)^\alpha \frac{\partial (x - Q)^\alpha}{\partial Q^\beta} + \frac{\partial U}{\partial Q^\beta}, \\ K - H &= -(p - P)^\alpha \frac{\partial (x - Q)^\alpha}{\partial t} + \frac{\partial U}{\partial t}. \end{aligned} \quad (6)$$

The distribution function in the new coordinates  $F(Q, P, s)$  is related to the distribution function in the old set of coordinates  $f(x, p, s)$  by

$$F(Q, P, s) DQDP = f(x, p, s) DxDP. \quad (7)$$

Equation (7) expresses particle conservation in the phase-space volume  $DxDP$  under the transformation given by Eq. (4). For a canonical transformation, the phase-space volume is conserved according to  $DxDP = DQDP$ , and therefore  $F(Q, P, s) = f[x(Q, P, s), p(Q, P, s), s]$ . The new distribution function satisfies the Vlasov equation  $dF/ds = 0$ . For a time-independent Hamiltonian, there exists a trivial solution to the Vlasov equation,  $F = G[K(Q, P)]$  for arbitrary function  $G$ . The periodic solution to the original Vlasov equation (1) can be found by inverting Eqs. (4) according to  $f(x, p, s) = G\{K_G[Q_G(x, p, s), P_G(x, p, s)]\}$ . Here, the subscript  $G$  denotes the implicit dependence of the solution on the choice of the function  $G$ . For solutions of this form, we can use Eq. (7) to express the original Hamiltonian in Eq. (2) as

$$\begin{aligned} H(x, p, s) &= \epsilon \left\{ \frac{p^\alpha p^\alpha}{2} + \frac{\kappa^\alpha(s) x^\alpha x^\alpha}{2} \right. \\ &\left. + \int L[x - x(\bar{Q}, \bar{P}, s)] G[K(\bar{Q}, \bar{P})] D\bar{Q} D\bar{P} \right\}. \end{aligned} \quad (8)$$

Equations (4) and (6) can be solved iteratively in terms of the small parameter  $\epsilon \ll 1$ . Specifically, we assume that

$$\begin{aligned} p &= P + \sum_{n=1} \epsilon^n p_n, \quad x = Q + \sum_{n=1} \epsilon^n x_n, \\ U &= \sum_{n=1} \epsilon^n U_n, \quad K = \sum_{n=1} \epsilon^n K_n, \end{aligned} \quad (9)$$

where  $p_n(Q, P, s)$ ,  $x_n(Q, P, s)$ ,  $U_n(Q, P, s)$  and  $K_n(Q, P, s)$  ( $n = 1, 2, \dots$ ) are functions to be determined by the iterative procedure.

Using Eqs. (9), we expand the function  $H$  in Eq. (8) according to

$$H(x, p, s) = \sum_{n=1} \epsilon^n H_n(Q, P, s). \quad (10)$$

Substituting the expansions [Eqs. (9) and (10)] into Eqs. (6), we obtain

$$\begin{aligned} x_n^\beta &= \ll \int ds \frac{\partial H_n}{\partial P^\beta} \gg \\ &+ \ll \int dt \sum_{l=1}^{n-1} \left( \frac{\partial p_{n-l}^\alpha}{\partial s} \frac{\partial x_l^\alpha}{\partial P^\beta} - \frac{\partial p_{n-l}^\alpha}{\partial P^\beta} \frac{\partial x_l^\alpha}{\partial s} \right) \gg, \\ p_n^\beta &= \bar{p}_n^\beta - \ll \int ds \frac{\partial H_n}{\partial Q^\beta} \gg \\ &- \ll \int ds \sum_{l=1}^{n-1} \left( \frac{\partial p_{n-l}^\alpha}{\partial s} \frac{\partial x_l^\alpha}{\partial Q^\beta} - \frac{\partial p_{n-l}^\alpha}{\partial Q^\beta} \frac{\partial x_l^\alpha}{\partial s} \right) \gg, \\ K_n &= \langle H_n \rangle - \sum_{l=1}^{n-1} \langle p_{n-l}^\alpha \frac{\partial x_l^\alpha}{\partial s} \rangle, \end{aligned} \quad (11)$$

where the average value  $\bar{p}_n$  satisfies the equation

$$\frac{\partial \bar{p}_n^\gamma}{\partial P^\beta} = \ll \sum_{l=1}^{n-1} \left( \frac{\partial p_{n-l}^\alpha}{\partial Q^\gamma} \frac{\partial x_l^\alpha}{\partial P^\beta} - \frac{\partial p_{n-l}^\alpha}{\partial P^\beta} \frac{\partial x_l^\alpha}{\partial Q^\gamma} \right) \gg. \quad (12)$$

Here,  $\langle a \rangle \equiv \int_0^T a(s) ds / S$  and  $\ll a \gg \equiv a - \langle a \rangle$ . For a prescribed Hamiltonian function  $H(x, p, s)$  [Eq. (8)], Eqs. (11) and (12) provide an iterative procedure which can be used to determine the canonical transformation in Eq. (4), and the new time-independent Hamiltonian  $K(P, Q)$  as implicit functions of the equilibrium function  $G$ .

## ILLUSTRATIVE APPLICATION

As a specific application, in this section we examine the canonical transformation in Eq. (4), valid up to fifth order in the small parameter  $\epsilon$ , for the intense beam system with Hamiltonian given by Eq. (8), for the choice of the lattice function  $\kappa^\alpha(s) = \bar{\kappa} \sin(\omega s) \eta^\alpha$ , with vector  $\eta = (1, -1)$ .

Here, we omit the details and only present the final results. The new time-independent Hamiltonian is determined to be [5]

$$K = \epsilon \left\{ \frac{P^\alpha P^\alpha}{2} \left( 1 + \epsilon^4 \frac{3\bar{\kappa}^2}{2\omega^4} \right) + \epsilon^2 \frac{\bar{\kappa}^2}{2\omega^2} \frac{Q^\alpha Q^\alpha}{2} \right. \\ \left. + \int D\bar{Q}D\bar{P}G(\bar{K}) \left[ L(Q - \bar{Q}) + \right. \right. \quad (13) \\ \left. \left. \epsilon^4 \frac{\bar{\kappa}^2}{4\omega^4} (Q - \bar{Q})^\alpha (Q - \bar{Q})^\beta \eta^\alpha \eta^\beta \frac{\partial^2 L}{\partial Q^\alpha \partial Q^\beta} \right] \right\}.$$

Furthermore, the detailed expressions for the canonical transformation are given by

$$x^\alpha = Q^\alpha + \epsilon^2 \frac{\bar{\kappa}}{\omega^2} \eta^\alpha Q^\alpha \sin(\omega s) \quad (14) \\ + 2\epsilon^3 \frac{\bar{\kappa}}{\omega^3} \eta^\alpha P^\alpha \cos(\omega s) + \epsilon^4 \left\{ -\frac{\bar{\kappa}^2}{8\omega^4} Q^\alpha \cos(2\omega s) \right. \\ \left. + \sin(\omega s) \frac{\bar{\kappa}}{\omega^4} \int D\bar{Q}D\bar{P}G \right. \\ \left. \times \left[ 2\delta_{\alpha\beta} + \frac{\partial}{\partial Q^\alpha} (Q - \bar{Q})^\beta \right] \eta^\beta \frac{\partial L}{\partial Q^\beta} \right\} + \epsilon^5 x_5^\alpha.$$

and

$$p^\alpha = P^\alpha + \epsilon \frac{\bar{\kappa}}{\omega} \eta^\alpha Q^\alpha \cos(\omega s) - \epsilon^2 \frac{\bar{\kappa}}{\omega^2} \eta^\alpha P^\alpha \sin(\omega s) \\ + \epsilon^3 \left\{ \frac{\bar{\kappa}^2}{4\omega^3} Q^\alpha \sin(2\omega s) + \frac{\bar{\kappa}}{\omega^3} \cos(\omega s) \right. \quad (15) \\ \left. \times \int D\bar{Q}D\bar{P}G \frac{\partial}{\partial Q^\alpha} \left( \eta^\beta (Q - \bar{Q})^\beta \frac{\partial L}{\partial Q^\beta} \right) \right\} \\ + \epsilon^4 \left\{ \frac{\bar{\kappa}^2}{8\omega^4} P^\alpha [12 + 5 \cos(2\omega s)] - \sin(\omega s) \frac{\bar{\kappa}}{\omega^4} \right. \\ \left. \times \int D\bar{Q}D\bar{P}G \frac{\partial}{\partial Q^\alpha} \left[ 2(P - \bar{P})^\beta \right. \right. \\ \left. \left. + P^\gamma \frac{\partial}{\partial Q^\gamma} (Q - \bar{Q})^\beta \right] \eta^\beta \frac{\partial L}{\partial Q^\beta} \right\} + \epsilon^5 p_5^\alpha.$$

Here, the expressions for  $x_5$  and  $p_5$  can be found in Ref. [5].

## DISCUSSION OF RESULTS

The results obtained by this method in Eq. (13)–(15) are consistent with previous results obtained to third order in the small parameter  $\epsilon = S\kappa_0^{1/2}$  in Refs. [2, 3]. Here, we have been able to extend the perturbative treatment to fifth order in the small parameter  $\epsilon$ , by avoiding the unnecessary calculation of the generating function as a function of a mixed set of variables. For a specific choice of distribution function  $G(K)$ , Eq. (13) can be solved to determine the new time-independent Hamiltonian  $K$ . The fifth-order corrections to the new Hamiltonian are of two kinds. The correction to the kinetic term gives a correction to the average frequency of the particle motion in the external oscillating field, whereas the last term gives the correction to the average self-field potential. The final term can be expressed as a self-field potential  $\Psi_m$  that satisfies the modified Poisson's equation

$$\Psi_m(Q) = \int L_m(Q - \bar{Q}) G(K) D\bar{Q}D\bar{P}, \quad (16)$$

with the modified Green's function  $L_m(Q)$  defined by

$$L_m(Q) = L(Q) + \epsilon^4 \frac{\bar{\kappa}^2}{4\omega^4} \left[ (Q^1)^2 \frac{\partial^2 L}{\partial (Q^1)^2} \right. \\ \left. + (Q^2)^2 \frac{\partial^2 L}{\partial (Q^2)^2} - 2Q^1 Q^2 \frac{\partial^2 L}{\partial Q^1 \partial Q^2} \right]. \quad (17)$$

Note that this new Green's function does not have cylindrical symmetry. However, it still possess quadrupolar symmetry.

Finally we note that the ordering assumed at the beginning of this paper, with  $P \sim 1$ , is not fully consistent with the final result. This can be seen from the fact that the average particle motion is on surfaces of constant average energy  $K = const.$ , and therefore, in general, we obtain  $P \sim \epsilon Q \sim \epsilon$ , while the initial assumption was that  $P \sim 1$ . The formulation of a more refined self-consistent ordering is being developed in Ref. [5].

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