Collective temperature anisotropy instabilities in intense charged particle beams\(^a\)^

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The classical electrostatic Harris instability and the electromagnetic Weibel instability, both driven by a large temperature anisotropy \(T_{\perp} / T_{\parallel} \ll 1\) that develops naturally in accelerators, are generalized to the case of a one-component intense charged particle beam with anisotropic temperature, including the important effects of finite transverse geometry and beam space-charge. Such instabilities may lead to an increase in the longitudinal velocity spread, which makes focusing the beam difficult, and may impose a limit on the beam luminosity and the minimum spot size achievable in focusing experiments. This paper describes recent advances in the theory and simulation of collective instabilities in intense charged particle beams caused by large temperature anisotropy. The new simulation tools that have been developed to study these instabilities are also described. Results of the investigations that identify the instability growth rates, levels of saturations, and conditions for quiescent beam propagation are also discussed. © 2007 American Institute of Physics. [DOI: 10.1063/1.2436847]

I. INTRODUCTION

Periodic focusing accelerators, transport systems, and storage rings\(^1\)–\(^3\) have a wide range of applications ranging from basic scientific research in high energy and nuclear physics, to applications such as ion-beam-driven high energy density physics and fusion, and spallation neutron sources. Of particular importance at the high beam currents and charge densities of practical interest are the effects of the intense self-fields produced by the beam space charge and current on determining the detailed equilibrium, stability, and transport properties. Charged particle beams confined by external focusing fields represent an example of non-neutral plasma.\(^4\) A characteristic feature of such plasmas is the non-uniformity of the equilibrium density profiles and the non-linearity of the self-fields, which makes detailed analytical investigation difficult. The development and application of advanced numerical tools such as eigenmode codes\(^5\)–\(^7\) and Monte-Carlo particle simulation methods\(^8\)–\(^13\) are often the only tractable approach to understand the underlying physics of different instabilities familiar in electrically neutral plasmas. Two such instabilities are the electrostatic Harris instability\(^,5\)–\(^9\),\(^11\),\(^14\) and the electromagnetic Weibel instability,\(^7\),\(^15\) both driven by a large temperature anisotropy that develops naturally in accelerators. The beam acceleration causes a large reduction in the longitudinal temperature. Indeed, for particles with charge \(e_i\) and mass \(m_i\) accelerated by a voltage \(V\) (here \(i\) denotes the initial state before acceleration, and \(f\) denotes the state after acceleration), the energy spread of particles in the beam does not change, and (non-relativistically) \(\Delta E_{bi} = m_i \Delta v_{bi}^2 / 2 = \Delta E_{bf} = m_i V_i \Delta v_{bf}\), where \(V_i = (2 e_i V / m_i)^{1/2}\) is the average beam velocity after acceleration. Therefore, the velocity spread-squared, or equivalently, the effective temperature, changes according to (for a non-relativistic beam) \(T_{bf} = T_{bi} (T_{bf} / 2 e_i V)\). For particles accelerated to highly relativistic energies \((\gamma_b \geq 1)\), \(T_{bf} = \left(\Delta p_{bf}\right)^2 / 2 m_b = c (\Delta p_{bf})^2 = c \gamma_b (\Delta p_{bf})\), where \(L\) and \(B\) denote laboratory and beam-frame quantities, respectively. The longitudinal temperature is proportional to momentum-spread-squared in the beam frame, \(T_{bf} = (T_{bi} / m_i c^2 \gamma_b^2 T_{bi}) T_{bf}\), whereas the transverse temperature remains the same, \(T_{\perp,bf} = T_{\perp,bi}\). As a result, the temperature anisotropy ratio after acceleration, \(T_{bf} / T_{\perp,bf} = T_{bf} / 2 e_i V\) (non-relativistic) or \(T_{bf} / T_{\perp,bf} = (T_{bf} / m_i c^2 \gamma_b^2)\) (relativistic), can become very small. This reduction in longitudinal temperature provides the free energy to drive collective temperature anisotropy instabilities. Such instabilities may lead to a deterioration of the beam quality (emittance growth, halo particle production, etc.). These instabilities may also lead to an increase in the longitudinal velocity spread, which will make focusing the beam difficult, and may impose a limit on the beam luminosity and the minimum spot size achievable in focusing experiments.

There is a significant amount of literature dedicated to the study of collective instabilities due to temperature anisotropy in intense charged particle beams. The electrostatic Harris instability has been studied theoretically for beams with a Kapchinskij-Vladimirskij (KV) distribution,\(^16\),\(^17\) and for a two-temperature Maxwellian distribution,\(^8\),\(^9\) and also computationally using particle-in-cell simulations.\(^18\)–\(^22\) The early numerical studies of this instability used the electrostatic particle-in-cell (PIC) code WARP, which is sufficiently noisy that resolving the linear stage of the instability with sufficient accuracy is difficult. Our previous numerical studies of the Harris instability used the eigenmode code bEASi,\(^5\) and the \(\delta f\) particle-in-cell code BEST,\(^8\),\(^9\),\(^11\) and allowed us to investigate both the linear and nonlinear stages of the Harris instability in considerable detail. The eigenmode code bEASi has also been used to study the linear stage of the electromagnetic Weibel instability.\(^7\)

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This paper reviews recent advances in the theory and simulation of collective instabilities in intense charged particle beams caused by large temperature anisotropy. We also describe new simulation tools that have been developed to study these instabilities. Results of the investigations that identify the instability growth rates, levels of saturations, and conditions for quiescent beam propagation are also discussed.

The organization of this paper is the following. In Sec. II and III, we discuss the physical mechanism for the electrostatic Harris and electromagnetic Weibel instabilities in intense charged particle beams with large temperature anisotropy. The beam eigenmode and spectra (bEAST) code developed to study linear properties of these instabilities is described in Sec. IV. The nonlinear δf beam equilibrium, stability, and transport (BEST) code has recently been updated to include the electromagnetic Darwin model, and is briefly described in Sec. V. In Sec. VI, we summarize several important simulation results obtained using the linear code bEAST and the nonlinear δf code BEST. Finally, conclusions are summarized in Sec. VII.

II. THE ELECTROSTATIC HARRIS INSTABILITY

For simplicity, the subsequent analysis is carried out in the beam frame (\(V_b=0\)). The laboratory frame expressions for frequencies and growth rates and their dependence on wave number can be readily obtained by applying the Lorentz transformation to the frequencies and wave numbers in the beam frame. In what follows, it is convenient to introduce the effective depressed betatron frequency \(\bar{\omega}_{\beta \perp}\) defined by

\[
\bar{\omega}_{\beta \perp}^2 = \frac{2 T_{\perp b}}{m_b r_{bh}^2} = \omega_j^2 - \bar{\omega}_{pb}^2/2, \tag{1}
\]

where \(T_{\perp b}\) is the transverse beam temperature, \(r_b\) is the root-mean-square beam radius, \(e_b\) and \(m_b\) are the charge and mass of a beam particle, and

\[
\bar{\omega}_{pb}^2 = \frac{4 \pi e_b^2}{m_b r_{bh}^2} \int_0^{\infty} dr n_b(r) \tag{2}
\]

is the average beam plasma frequency-squared, where \(n_b(r)\) is the radial density profile of the beam particles and \(r_{bh}\) is the radius of the perfectly conducting wall. The normalized tune depression \(\bar{\nu}/\nu_0\) is defined by

\[
\frac{\bar{\nu}}{\nu_0} = \frac{\omega_{\beta \perp}}{\omega_j}, \tag{3}
\]

where \(\omega_j=\text{const}\) is the transverse frequency associated with the applied focusing field in the smooth-focusing approximation.

We now briefly illustrate the physical mechanism for the electrostatic Harris instability in intense particle beams. As shown in previous studies, the dipole mode has the highest growth rate, and for \(T_{\parallel b}=0\) the growth rate is an increasing function of \(k r_b\) and approaches a maximum value for \(k r_b^2 \gg 1\). Therefore, we consider dipole-mode perturbations with \(k r_b^2 \gg 1\), which in lowest order correspond to a displacement of the beam charge mainly along the beam propagation direction, arranged as a dipole-mode perturbation as shown in Fig. 1. One can distinguish three possibilities: \(\bar{\omega}_{pb} \gg \omega_{\beta \perp}, \bar{\omega}_{pb} \ll \omega_{\beta \perp}\), and \(\bar{\omega}_{pb} \approx \omega_{\beta \perp}\). If \(\bar{\omega}_{pb} \gg \omega_{\beta \perp}\), the charge oscillation will be mostly along the beam propagation direction due to the electrostatic restoring force, and as a result, the mode frequency will be close to the plasma frequency \(\omega = \bar{\omega}_{pb}\). In the opposite limit, when \(\bar{\omega}_{pb} \ll \omega_{\beta \perp}\), the charge perturbations will oscillate with the frequency \(\omega_{\beta \perp}\), mainly in the direction transverse to the beam propagation direction due to the restoring betatron force. In this case, the mode frequency will be close to the average betatron frequency \(\bar{\omega}_{\beta \perp}\). Finally, if \(\bar{\omega}_{pb} \approx \omega_{\beta \perp}\), the charge perturbation, moved longitudinally by the electrostatic restoring force, will at the same time traverse the beam transversely. In this case, the arrangement of the charge perturbation does not change, and the mode will have approximately zero frequency with \(\text{Re} \omega=0\). If one now examines the motion of an individual particle in this unchanging dipole electric field \(E_x \sim x_{\perp}\), with longitudinal acceleration \(d^2 x_{\parallel}/dt^2 = E_z \sim x_{\perp} \cos(\omega_{\beta \perp} t)\), one finds that during one-half period of oscillation each particle will acquire a longitudinal displacement \(\Delta z\) that is opposite to the direction of the electric field at this point in space. This means that the particle will move toward the excess charge, and therefore the charge perturbation will be enhanced, which will result in instability with \(\text{Im} \omega > 0\). One can also see the effects that longitudinal temperature has on the instability. If, during one-half period of transverse oscillation \(\pi/\omega_{\beta \perp}\), a particle with average speed \(v_{\|}\) travels a distance larger than the half-wavelength of the perturbation \(\lambda/2\), then the perturbation enhancement shown in Fig. 1 will not occur, and the instability is absent. This provides the threshold condition for the onset of instability, i.e.,

\[
\lambda_z \geq v_{\|}\frac{2\pi}{\omega_{\beta \perp}} \Rightarrow \frac{T_{\parallel b}}{T_{\perp b}} \leq \frac{\omega_{\beta \perp}}{2 k^2 T_{\perp b}}, \tag{4}
\]

where use has been made of Eq. (1). Since we have assumed \(k^2 r_b^2 \gg 1\), Eq. (4) implies that the threshold for instability satisfies \(T_{\parallel b}/T_{\perp b} \ll 1\). We can now summarize the necessary
III. THE ELECTROMAGNETIC WEIBEL INSTABILITY

Another instability known from the study of electrically neutral plasmas that is also driven by temperature anisotropy is the electromagnetic Weibel instability. The filamentation instability of intense charged particle beams propagating in neutralizing background plasma is a similar instability, and is also often called the Weibel instability. The mechanism for the Weibel instability is illustrated in Fig. 2. An initial current perturbation creates a magnetic-field perturbation that in turn acts through the v × B/c force on particles moving with characteristic thermal velocity to displace them in the direction shown in Fig. 2, which enhances the initial current perturbation. This results in a purely growing perturbation with real frequency Re ω > 0. Since the driving force is magnetic, the instability is weak, with growth rate proportional to v₀₀/c. For the case of a one-component beam, the finite transverse geometry of the beam makes a detailed analytical description difficult. We provide here a simple physical model based mainly on the results of simulations, which will be presented later.

We consider a charged particle beam confined inside a circular conducting pipe of radius r₀ by an external linear force F = −m₀β₀²x₀. In the smooth-focusing approximation, for simplicity, the analysis is carried out in the beam frame (V₀ = 0). The beam is confined in the transverse direction provided β₀²/2α₀² < 1. Here, β₀² = 4πε₀²n₀/r₀ is the on-axis (r = 0) plasma frequency-squared, and α₀ is the average oscillation frequency of a beam particle with mass m₀ and charge e₀ in the applied focusing field. It follows from the numerical studies presented later in this paper that the fastest growing modes correspond to rigid rotations of the beams slices with δJ₀ ∼ r for β₀²/2α₀² → 1. The growth rate is an increasing function of k₀r₀ and approaches a maximum value for k₀r₀² ≫ 1. Therefore, in leading order, the perturbed magnetic field is given approximately by δB ≈ ik₀δA, for k₀r₀² ≫ 1. From Maxwell’s equations it follows that

$$\delta A_θ(x, t) = \frac{r}{rb} \exp[i(kz - \omega t)].$$

The longitudinal equation of motion for a beam particle becomes

$$\ddot{z} = -\frac{e₀v₀}{m₀} \frac{\dot{A}_θ r(t)\dot{v}_θ(t)}{cr₀} \exp[i(kz₀ - \omega t)].$$

In the smooth-focusing approximation, the unperturbed motion is in a cylindrically symmetric potential U(r), and therefore the angular momentum is conserved, i.e., r(t)\dot{v}_θ(t) = const. Integrating Eq. (6) with respect to time t, we obtain

$$z(t) = \frac{k_0}{\omega} e₀ r(t) \dot{v}_θ(t) \exp[i(kz₀ - \omega t)] = \frac{v_0 e₀ \delta B_z}{\omega m₀ c}.$$  

(7)

The average axial displacement is given by \langle z \rangle ∼ \langle v_θ \rangle = 0, and therefore the density perturbation \dot{δn}_θ = -δn_0 \ddot{z}/\ddot{z} is zero. Therefore, for the current perturbation, we obtain

$$\frac{\partial \delta J_θ}{\partial t} + \frac{\partial}{\partial z} \left( \frac{\dot{A}_θ n_0 \ddot{v}_θ}{\partial t} \right) = 0,$$  

(8)

$$\delta J_θ = -e₀ \delta n_0 \ddot{v}_θ \frac{\partial}{\partial z} = -e₀ \delta n_0 \ddot{v}_θ \left( \frac{v_0}{m₀ c} \right)^2 \frac{\partial}{\partial z}.$$  

Substituting Eq. (8) into Maxwell’s equation \partial B_z/\partial z = 4πδJ_θ/c, we obtain the simple dispersion relation

$$1 = -\frac{\omega_0^2}{\omega^2} \left( \frac{v_0^2}{c^2} \right).$$  

(9)

where \omega_0² = 4πε₀²n₀/m₀ is the average beam plasma frequency squared. Noting that T⁻¹ = m₀(v₀² + v₀²)/2 = m₀(v₀²), we can express the growth rate in this simple model of the Weibel instability as

$$\gamma = \omega_0 \sqrt{\frac{T_{-1}}{m₀ c^2}} = \omega_0 \sqrt{\frac{T_{-1}^b}{m₀ c^2}} = 0.71 \omega_0 \frac{v_{θb}^b}{c},$$  

(10)

where v_{θb} = v_{θb}/\sqrt{2} T_{-1}/m₀ is the transverse thermal velocity.

IV. DESCRIPTION OF BEAM EIGENMODE AND SPECTRA (bEAST) CODE

For an arbitrary equilibrium distribution, one cannot solve the stability problem analytically and must employ numerical techniques. To investigate stability properties numerically, we make use of the linear eigenmode method, which searches for the roots of the matrix dispersion relation, as implemented in the beam eigenmode and spectra (bEAST) code.

The bEAST code assumes small-amplitude electrostatic perturbations of the form
\[ \delta \phi(x,t) = \hat{\delta} \phi(r) \exp(i m \theta + i k z - i \omega t), \]  
(11)

where \( \delta \phi(x,t) \) is the perturbed electrostatic potential, \( k \) is the axial wave number, \( m \) is the azimuthal mode number, and \( \omega \) is the complex oscillation frequency, with \( \text{Im} \ \omega > 0 \) corresponding to instability (temporal growth). We also assume that the beam is located inside a perfectly conducting cylindrical pipe with radius \( r_w \). Electromagnetic perturbations are assumed to be of the form

\[ \delta A_b(x,t) = \hat{\delta} A(r) \exp[i(kz - \omega t)]. \]  
(12)

All perturbations are about the thermal equilibrium distribution with temperature anisotropy \( (T_{\perp b} > T_{\parallel b}) \) described in the beam frame \( (V_b=0 \ and \ \gamma_b=1) \) by the self-consistent axisymmetric Vlasov equation

\[ f_b^0(r,p) = \frac{\hat{n}_b}{(2\pi m_b)^{3/2}} T_{\perp b}^{-1/2} \exp \left[ -\frac{H_{\perp b}}{T_{\perp b}} \right] \frac{p_r^2}{2m_b T_{\perp b}} \]  
(13)

Here, \( H_{\perp b} = \frac{p_r^2}{2m_b} + \sum_{\lambda} \alpha_{\lambda} \phi_{\lambda}(r) \), where \( \alpha_{\lambda} \) are constants, and the complete set of vacuum eigenfunctions \( \{ \phi_{\lambda}(r) \} \) is defined by \( \phi_{\lambda}(r) = \sum_{\lambda} A_{\lambda} J_{\lambda}(\lambda r/r_w) \). Here, \( \lambda_{\lambda}^2 \) is the \( \lambda \)th zero of \( J_{\lambda}(\lambda_{\lambda} r/r_w) = 0 \), and \( A_{\lambda} = \sqrt{2/J_{\lambda+1}(\lambda_{\lambda})} \) is a normalization constant such that \( \int_0^1 r dr \phi_{\lambda}(r) \phi_{\lambda}(r) = \delta_{\lambda,\lambda} \). Electromagnetic perturbations are also expanded in terms of the complete set of vacuum eigenfunctions \( \delta A_b(r) = \sum_{\lambda} \alpha_{\lambda} A_{\lambda}(r) \), where \( A_{\lambda}(r) = A_{\lambda} J_{\lambda}(\lambda r/r_w) \) and \( J_{\lambda}(\lambda_{\lambda} r/r_w) = 0 \). Using the method of characteristics, analysis of the linearized Vlasov-Maxwell equations leads to an infinite-dimension matrix dispersion equation,\(^5\)–\(^9\),\(^11\),\(^15\)

\[ \sum \alpha_{\lambda} D_{n,m}(\omega) = 0, \]  
(14)

where the elements of the dispersion matrix \( D_{n,n'}(\omega) \) are defined by

\[ D_{n,n'}^{el}(\omega) = \frac{J_{\lambda_{\lambda}+1}(\lambda_{\lambda}^2)}{2} \left[ \lambda_{\lambda}^2 + k_z^2 r_w^2 - \frac{\omega^2}{c^2} \right] \delta_{n,n'} + \chi_{n,n'}^{el}(\omega) \]  
(15)

for electrostatic perturbations, and by

\[ D_{n,n'}^{em}(\omega) = \frac{J_{\lambda_{\lambda}+1}(\lambda_{\lambda}^2)}{2} \left[ \lambda_{\lambda}^2 + k_z^2 r_w^2 - \frac{\omega^2}{c^2} \right] \delta_{n,n'} + \chi_{n,n'}^{em}(\omega) \]  
(16)

for electromagnetic perturbations. Here, \( \chi_{n,n'}^{el} \) is the beam-induced susceptibility and is defined by

\[ \chi_{n,n'}^{el}(\omega) = \frac{\hat{n}_b}{\lambda_{\lambda}^2 q_{n,n'} + \int_0^\infty ds \exp \left[ i \omega s - \frac{s^2}{2m_b} T_{\perp b} \right] - i \omega + \left[ 1 - \frac{T_{\perp b}}{T_{\perp b}} \right] \frac{k_z^2 T_{\perp b}}{2m_b} \} Q_{n,n'}^{el}(s) \]  
(17)

for electrostatic perturbations,\(^5\)–\(^8\),\(^9\),\(^11\),\(^14\),\(^17\)

\[ Q_{n,n'}^{el}(s) = \frac{1}{m_b \lambda_{\lambda}^2} \sum_p \int \frac{dP dH_{\perp b}}{\omega_r T_{\perp b}} \exp \left[ -\frac{H_{\perp b}}{T_{\perp b}} \right] \frac{(p_{n,n'}^{em})^* p_{n,n'}^{em}}{2m_b} \exp[ -i \omega (\rho \omega_r + m \omega \omega_b) ] \]  
(18)

In Eqs. (17) and (18), \( q_{n,n'} \) and \( p_{n,n'}^{em} \) are defined by

\[ q_{n,n'} = \int_0^1 dx x N(x) J_m(\lambda_{\lambda}^2 x) J_m(\lambda_{\lambda}^2 x), \]  
(19)

and the orbit integral \( p_{n,n'}^{em} \) is defined by

\[ p_{n,n'}^{em}(H_{\perp b}, P_b) = \int_0^{T_{\perp b}} d\tau \int_{r_w}^{r_{T_{\perp b}}} \frac{\lambda_{\lambda}^2(\tau)}{r_w} \exp[ -i \rho \omega_r \tau + i m \omega \omega_b \tau]. \]  
(20)

The beam-induced susceptibility for low-frequency electromagnetic modes with \( \omega \ll \omega_b \) is defined by

\[ \chi_{n,n'}^{em}(\omega) = \frac{\hat{n}_b}{c^2} \left[ 1 + \frac{\omega}{m \omega_b} n_{mb}(\omega) \right] Q_{n,n'}^{em}(\omega), \]  
(21)

where

\[ Q_{n,n'}^{em}(\omega) = \int \frac{dP dH_{\perp} dH_{\parallel}}{m_b \omega_r T_{\perp b}} \exp \left[ -\frac{H_{\perp b}}{T_{\perp b}} \right] (I_n)_{\perp} \]  
(22)

The orbit integral \( I_n \) is defined by

\[ I_n(H_{\perp b}, P_b) = \int_0^{T_{\perp b}} d\tau \int_{r_w}^{r_{T_{\perp b}}} \left[ \lambda_{\lambda}^2(\tau) \right] \frac{r_{T_{\perp b}}}{r_w} \]  
(23)

Here, \( P_b \) is the canonical angular momentum, \( \omega_b = 4 \pi e^2 n_b/m_b \) is the on-axis plasma frequency-squared, \( \lambda_{\lambda}^2 = \frac{T_{\perp b}}{4 \pi \omega_b^2} n_b \) is the perpendicular Debye length-squared, and \( v_{mb} = \sqrt{2 T_{\perp b} / m_b} \). In the orbit integrals in Eqs. (20) and (23), \( r \) and \( \theta \) are the transverse orbits in the equilibrium field configuration such that \( \theta(0)=0 \), \( r(0)=r_{\min}(H_{\perp b}, P_b) \) is the minimum radial excursion of the particle trajectory undergoing periodic motion with frequency \( \omega_b(H_{\perp b}, P_b) = 2 \pi / T_r \), and \( \omega_b(H_{\perp b}, P_b) = \theta(T_r) / T_r \) is the average frequency of angular rotation. In Eq. (18), \( (\cdot)^* \) denotes complex conjugate, and \( N(x) = n_{mb}^{1/2}(x_{mb}) / \hat{n}_b \) is the normalized density profile, where \( n_{mb}^{1/2}(r) = \int d^3 \phi d^3 p \) is the normalized density profile.

The beam eigenmode and spectra (bEAST) code solves Eq. (14) in several steps. First, the particle orbits \( r(\tau) \) and \( \theta(\tau) \) in the equilibrium field configuration are calculated for one complete oscillation period \( T_r \), and the frequencies \( \omega_b(H_{\perp b}, P_b) \) and \( \omega_b(H_{\perp b}, P_b) = \theta(T_r) / T_r \) are obtained. Next, a fast Fourier transform (FFT) is used to calculate the orbit integrals in Eqs. (20) and (23). Then, in the next step, the matrices \( Q_{n,n'}^{el}(s) \) [Eqs. (18) and (22)] and \( q_{n,n'} \) [Eq. (19)] are calculated, stored, and then used repeatedly to recalculate the beam-induced susceptibility [Eqs. (17) and (21)] and dispersion matrix [Eqs. (15) and (16)] during the search for the eigenvector of the dispersion matrix \( D_{n,n}(\omega) \) [Eq. (14)] with zero eigenvalue. Note that the matrices \( Q_{n,n'}^{el}(s) \) and \( q_{n,n'} \) are calculated only once, thanks to the separation of the particle variables \( (H_{\perp b}, P_b, r, \theta) \) from the dispersion equation variables \( \omega \) and \( k_z \) in Eq. (17). The typical number of particle
trajectories used in the calculations is 300, with 16 time steps during one oscillation period $T_r$, which is significantly less than the number of particles and time steps used in PIC simulations.\textsuperscript{8,9,11} The method described here works well for finding the unstable modes, or slightly damped modes. For highly damped modes, an accurate integration in Eq. (17) requires calculation of the matrix $Q_{n,n'}^{T} \langle s \rangle$ of values of $s > \text{Im} \omega/(k^2 T_r/m_p)$, which can be demanding computationally.

V. DESCRIPTION OF BEAM EQUILIBRIUM, STABILITY, AND TRANSPORT (BEST) CODE

To investigate the nonlinear stage of instability, we make use of the nonlinear $\delta f$ method\textsuperscript{29} described below, as implemented in the beam equilibrium, stability, and transport (BEST) code.\textsuperscript{1,10,13} This code has recently been extended to include electromagnetic phenomena in electrically neutral plasmas. As it turns out, the original Darwin Lagrangian that described electromagnetic interactions of charged particles accurate up to second order in $u/c$, where $u$ is the particle velocity and $c$ is the speed of light. The Lagrangian Darwin model was later reformulated as a model with its own set of field equations and applied to study low-frequency electromagnetic phenomena in electrically neutral plasmas. As it turns out, the original Darwin Lagrangian model is equivalent to neglecting the transverse part of the displacement current in Ampere’s law. As a key consequence, high-frequency light waves are eliminated from the Maxwell-Vlasov system. This greatly relaxes the time-step restrictions for numerical simulations, and it avoids the Courant condition $\Delta x/c \Delta t < 1$. The resulting Maxwell equations are elliptic and depend only on instantaneous particle quantities. Also, since high-frequency light waves are eliminated, the simulation noise is greatly reduced. The reduced Maxwell equations for the Darwin model used in the modified BEST code can be expressed as

$$\nabla \cdot \mathbf{E}^T = 4\pi \rho,$$  \hspace{1cm} (24)

$$\nabla \times \mathbf{B} = (4\pi c) \mathbf{J}^T,$$  \hspace{1cm} (25)

$$\nabla \times \mathbf{E}^T = -(1/c) \partial \mathbf{B}/\partial t,$$  \hspace{1cm} (26)

$$\nabla \cdot \mathbf{B} = 0,$$  \hspace{1cm} (27)

where

$$\mathbf{E}^T = -\nabla \phi,$$  \hspace{1cm} (28)

$$\mathbf{J}^T = \mathbf{J} - \frac{1}{4\pi} \frac{\partial \nabla \phi}{\partial t},$$  \hspace{1cm} (29)

and

$$\nabla \times \mathbf{E}^T = 0, \quad \nabla \cdot \mathbf{E}^T = 0, \quad \nabla \cdot \mathbf{J}^T = 0.$$  \hspace{1cm} (30)

Expressing $\mathbf{B} = \nabla \times \mathbf{A}$, and using the Coulomb gauge with $\nabla \cdot \mathbf{A} = 0$, Ampere’s equation takes the form

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}^T,$$  \hspace{1cm} (31)

where

$$\mathbf{E}^T = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$  \hspace{1cm} (32)

For comparison, the original Ampere’s law takes the form

$$\nabla^2 \mathbf{A} = -(1/c^2) \partial^2 \mathbf{A}/\partial t^2 = -\frac{4\pi}{c} \mathbf{J}^T.$$  \hspace{1cm} (33)

However, as noted previously,\textsuperscript{30} the presence of the time derivative of the vector potential in the equations of motion $d\mathbf{p}/dt = -(q/c) \partial \mathbf{A}/\partial t + \cdots$ can cause numerical instabilities in particle simulations because of the time-centering problem in particle pushing. These difficulties are avoidable if we introduce the canonical momentum $\mathbf{P} = \mathbf{p} + (q/c) \mathbf{A}$.\textsuperscript{30} Specifically, the equations of motion become

$$\frac{d\mathbf{x}}{dt} = \mathbf{v},$$  \hspace{1cm} (33)

$$\frac{d\mathbf{P}}{dt} = \frac{q}{c} \nabla (\mathbf{v} \cdot \mathbf{A}) - q \nabla \phi - m \omega_p^2 \mathbf{x}.$$  \hspace{1cm} (34)

Here

$$\mathbf{v} = \mathbf{p}/m \gamma, \quad \gamma = [1+(p/mc)^2]^{1/2}, \quad \mathbf{p} = \mathbf{P} - \frac{q}{c} \mathbf{A}.$$  \hspace{1cm} (35)

Thus, by transforming variables from mechanical momentum $\mathbf{p}$ to canonical momentum $\mathbf{P}$, the time derivative of $\mathbf{A}$ conveniently disappears from the equations of motion. To calculate the particle trajectories in Eqs. (33) and (34), one needs to calculate only the electrostatic potential $\phi$ and the electromagnetic vector potential $\mathbf{A}$. The Vlasov equation in the new variables can be expressed as

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + \frac{d\mathbf{x}}{dt} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} + \frac{d\mathbf{P}}{dt} \frac{\partial \mathbf{F}}{\partial \mathbf{P}} + \frac{\partial \mathbf{F}}{\partial \mathbf{P}} = 0,$$  \hspace{1cm} (36)

where the characteristics are defined by Eqs. (33) and (34). The electrostatic potential $\phi$ is determined from Poisson’s equation

$$\nabla^2 \phi = -4\pi \rho,$$  \hspace{1cm} (37)

and the electromagnetic vector-potential $\mathbf{A}$ is determined by solving a system of coupled equations of the Helmholtz type, i.e.,

$$\left( \nabla^2 - \frac{\omega_p^2}{c^2} \right) \mathbf{A} + \nabla \psi = -\frac{4\pi}{c} \mathbf{J}_p,$$  \hspace{1cm} (38)

$$\nabla \cdot \mathbf{A} = 0.$$  \hspace{1cm} (39)

Here the factor $\omega_p^2/c^2 = (4\pi n_0^2/mc^2)$ is $d^3 P F/\gamma$ arises from introducing the canonical momentum, the potential $\psi$ formally solves $\nabla^2 \psi = -(4\pi/c) \nabla \cdot \mathbf{J}$, which removes the longitudinal part of the current, and the current $\mathbf{J}_p$ occurring in Eq. (38) is defined by
\[ J_p = q \int d^3 P \frac{P}{\gamma m} F. \] (40)

For the case of heavy ions with \( r_w^2 \omega_i^2 / c^2 \ll 1 \), the skin term can be neglected in Eq. (38), and the above system of equations is linear. For electrons, the skin term is not generally negligible, and the system is nonlinear and is solved by iteration. For a perfectly conducting cylindrical wall of radius \( r_w \), the boundary conditions for \( \phi, A, \) and \( \psi \) are especially simple, i.e.,

\[ \phi_{rw} = A_{rw} = \psi_{rw} = 0. \] (41)

The boundary condition for \( \psi \) follows from the fact that \( \psi = -(1/c) \partial \phi / \partial t \).

In the corresponding \( \delta f \) formalism, the solutions to the nonlinear Vlasov-Maxwell equations are expressed as \( F = F_0 + \delta F, \phi = \phi_0 + \delta \phi, A = A_0 + \delta A \), where \( (F_0, \phi_0, A_0) \) are known equilibrium solutions (\( \partial \phi / \partial t = 0 \)). The perturbed distribution is determined from

\[ \frac{d \delta f}{dt} = - \frac{\partial}{\partial x} \left( \delta F_0 \right) - \frac{\partial P}{\partial x} \frac{\partial}{\partial P} \delta F_0, \] (42)

where \( \delta \) denotes the perturbed particle trajectories obtained by using the perturbed potentials \( \delta \phi \) and \( \delta A \). In the particle simulations using the modified BEST code, the perturbed \( \delta f(x,p,t) \) is given by the weighted Klimontovich representation,

\[ \delta f = \sum_{i=1}^{N} w_i \delta(x-x_i) \delta(P-P_i). \] (43)

Here, \( N \) is total number of particles in the simulation. The weight function \( w \), defined by \( w = \delta f / F \), evolves according to

\[ \frac{dw}{dt} = (1-w) \frac{1}{F_0} \frac{d \delta f}{dt}. \] (44)

In the modified BEST code, the nonlinear particle simulations are carried out by iteratively advancing the particle motion, including the weights they carry, and updating the fields by solving the perturbed Maxwell’s equations with appropriate boundary conditions at the cylindrical, perfectly conducting wall at radius \( r_w \).

The \( \delta f \) approach is fully equivalent to the original nonlinear Vlasov-Maxwell equations, but the noise associated with representation of the background distribution \( F_0 \) in conventional particle-in-cell (PIC) simulations is removed. In the \( \delta f \) approach, the simulation particles are used to represent only a small part of the entire distribution \( \delta f = F - F_0 \), and therefore the statistical error in the simulations is proportional to \( \epsilon_{\delta f} \sim \delta \sqrt{N} \), whereas the error in PIC simulations is proportional to \( \epsilon_{\text{PIC}} \sim 1 / \sqrt{N} \). Therefore, the typical gain in accuracy in \( \delta f \) simulations compared to PIC simulations with the same number of particles is \( \epsilon_{\delta f} / \epsilon_{\text{PIC}} \sim \sqrt{N} \). This fact allows much more accurate simulations of the nonlinear dynamics and instability thresholds when \( |\delta \sqrt{N} | \ll 1 \). In addition, the \( \delta f \) method can be used to study linear stability properties, provided all nonlinear terms in the dynamical equations of motion are neglected. This corresponds to replacing the term \( 1-w \) with \( 1 \) in Eq. (44) for the weights, and moving the particles along the trajectories calculated in the unperturbed fields \( \phi_0 \) and \( A_0 \).

The \( \delta f \) method described above has been implemented in the three-dimensional electromagnetic particle-in-cell code (BEST) in cylindrical geometry with a perfectly conducting cylindrical wall at radius \( r_w \). Maxwell’s equations (37)–(39) are solved using fast Fourier transform (FFT) techniques in the longitudinal and azimuthal directions. The particle positions [Eqs. (33) and (34)] and weights [Eq. (44)] are advanced using a second-order predictor-corrector algorithm. The BEST code is parallelized using the message passing interface (MPI) with domain decomposition in the direction of beam propagation. The NetCDF data format is used for large-scale diagnostics and visualization. Typical simulation runs consist of \( 10^7 \) simulation particles and are performed on the IBM SP/RS 6000 at NERSC.

VI. NUMERICAL SIMULATIONS OF THE HARRIS AND WEIBEL INSTABILITIES

Detailed simulations of the electrostatic Harris instability using the eigencode mode bEASI and the nonlinear \( \delta f \) code BEST have been performed, and results can be found in Refs. 5, 6, 8, 9, and 11. Here, for completeness, we summarize some of the most important results.

Figure 3 shows plots of the normalized growth rate \((\text{Im} \omega)_{\text{max}} / \omega_i \) at maximum growth versus normalized tune depression \( \nu / \nu_0 \) for \( T_{\|} / T_{\perp} = 0 \) and azimuthal mode number \( m = 1 \) (dotted curve). Here, wall radius \( r_w = 3.5 \lambda_b \). The results are obtained using the eigencode mode bEASI.\(^{5,6}\) The thick solid curve corresponds to the simple theoretical estimate from Ref. 6. Only the \( m = 1 \) results are shown since it has the largest growth rate. The \( m = 1 \) dipole mode is purely growing with \( \Re \omega = 0 \) and \( (\text{Im} \omega)_{\text{max}} / \omega_i = 0.34 \) for \( \nu / \nu_0 = 0.62 \). Here, \( \omega_i \) is the betatron frequency due to the applied focusing field. Note that the instability is absent for \( \nu / \nu_0 > 0.82 \). Therefore, the Harris instability is absent for beams with sufficiently small tune shift \( \delta \nu / \nu_0 = (\nu - \nu_0) / \nu_0 \). Figure 4 shows the threshold value of the anisotropy \( T_{\|} / T_{\perp} \) as a function of the normal-
ized tune depression $\bar{v}/v_0$ obtained using the eigenmode code bEASt.\textsuperscript{5,6} Note from Fig. 4 that the maximum threshold value, $T_{\|b}^{th}/T_{\perp b}=0.11$, is achieved for moderately intense beams with $\bar{v}/v_0=0.4$. The curves in Fig. 4 are results obtained with the bEASt code, and the dots correspond to the longitudinal beam temperature $T_{\|b}=m_0(\nu_0^2)$ obtained with the nonlinear $\delta f$ BEST code after the instability saturates. As shown in Refs. 5 and 6, the Harris instability saturates non-linearly by particle trapping and quasilinear relaxation.

We now present typical numerical results for the Weibel instability obtained using the eigenmode code bEASt, and the linearized version of the $\delta f$ code BEST, for the case in which $r_0=3r_{avg}$, $T_{\|b}/T_{\perp b}=0$, $\partial/\partial \theta=0$. Figure 5 shows plots of the normalized growth rate $(\text{Im} \omega)/(\omega^2 r_0/c)$ versus $k_z r_b$ obtained for normalized skin depth $c/r_0 \hat{\omega}_{ph}=100$ and several values of the normalized depressed tune $\bar{v}/v_0 =0.09(4), 0.4(3), 0.72(2)$. Also shown are the results of a linear simulation using the Darwin BEST code for $\bar{v}/v_0 =0.88, 0.92, 0.95, 0.97(1)$, $c/r_0 \hat{\omega}_{ph}=10$, and $T_{\|b}/T_{\perp b}=10^{-4}$.

Numerical studies using the eigenmode code bEASt have shown\textsuperscript{7} that the results are insensitive to the normalized skin depth provided $c/r_0 \hat{\omega}_{ph}$ $\approx$ 1. As evident from Fig. 5, the results obtained with the $\delta f$ code BEST are consistent with the results obtained using the eigenmode code bEASt. Plots of the normalized maximum growth rate $(\text{Im} \omega)_{max}/(\omega^2 r_0/c)$ versus the average depressed tune $\bar{v}/v_0$ for $T_{\|b}/T_{\perp b}=0$ and $c/r_0 \hat{\omega}_{ph}=100$ obtained using the bEASt code are shown in Fig. 6. The maximum growth rate is achieved for moderately intense beams with $\bar{v}/v_0 =0.73$. The dots are results of linear runs using the Darwin BEST code for $T_{\|b}/T_{\perp b}=10^{-4}$ and $c/r_0 \hat{\omega}_{ph}=10$. Again, the results obtained using both codes are in good agreement. Figure 7 shows the normalized longitudinal threshold temperature $(T_{\|b}/T_{\perp b})c^2/r_0^2 \hat{\omega}_{ph}^2$ for the onset of instability plotted versus the normalized tune depression $\bar{v}/v_0$ for normalized skin depth $c/r_0 \hat{\omega}_{ph}=100$.

The nonlinear stage of the Weibel instability is illustrated in Figs. 8–11 for a beam with $\bar{v}/v_0 =0.88$ and initial temperature ratio $T_{\|b}/T_{\perp b}=10^{-4}$ and $c/r_0 \hat{\omega}_{ph}=10$. Figure 8 shows a plot of the effective longitudinal temperature $T_{\|b}=m_0(\nu_0^2)$ normalized to the initial longitudinal temperature $T_{\|b}^{th}$, as a function of time. The initial linear stage is followed by a stage where the temperature grows superexponentially due to particle trapping. At later stages, the temperature varies
slightly due to quasilinear mixing. Figure 9 shows plots of the $z$-averaged longitudinal velocity distribution at $t=0$, and at a time after saturation. After saturation, the longitudinal velocity distribution remains nearly Maxwellian. Figure 10 shows the time history of the electrostatic potential $\frac{e_b \delta \phi}{m_b}$ and the azimuthal component of the vector potential $\frac{e_b v_{th} A}{m_b c}$ are plotted versus time for a beam with $\bar{\nu}/\nu_0=0.88$ and initial temperature ratio $T_{th}/T_{lb}=10^{-4}$ and $c/r_b \omega_{pe}=10$.

VII. CONCLUSIONS

To summarize, we have generalized the analysis of the classical Harris and Weibel instabilities to the case of a one-component intense charged particle beam with anisotropic temperature. For a long, coasting beam, the delta-f particle-in-cell code BEST and the eigenmode code bEASt have been used to determine the detailed 3D stability properties over a wide range of temperature anisotropy and beam intensity. It has been shown that intense beams with $\bar{\nu}/\nu_0<0.82$ and $T_{th}/T_{lb}<0.11$ are linearly unstable to electrostatic perturbations (Harris-type instability). The instability is kinetic in nature and is due to the coupling of the particles’ transverse betatron motion with the longitudinal plasma oscillations excited by the perturbation. It has also been shown that finite transverse geometry introduces the Weibel instability threshold $T_{th}/T_{lb}=10^{-0.7} (\omega_{pe} c^2 / e_b)$ $v_{th}^2 / c^2$. This makes the Weibel instability much less dangerous for intense beams with normalized tune $\bar{\nu}/\nu_0<0.82$. This is because such intense beams are unstable due to the electrostatic Harris instability, which saturates at much larger longitudinal temperature, $(T_{th}/T_{lb})^{\text{Weibel}} < (T_{th}/T_{lb})^{\text{Harris}} = 0.1$, and has
much larger growth rate, \( \gamma_{\text{Weibel}} / \gamma_{\text{Harris}} \sim v_0^{\text{th}} / c \ll 1 \). Therefore, the electromagnetic Weibel instability is likely to be an important instability mechanism in relativistic one-component charged particle beams with \( \bar{v} / v_0 > 0.82 \), but not in intense beams with \( \bar{v} / v_0 < 0.82 \). To study the nonlinear stage of the Weibel instability, the electromagnetic Darwin model has been implemented in the \( \delta f \) particle-in-cell code BEST. The results of the nonlinear simulations show that the nonlinear saturation is governed by longitudinal particle trapping (electrostatic trapping for the Harris instability and electromagnetic trapping for the Weibel instability). Even though in Secs. IV–VI we have chosen the initial longitudinal momentum distribution to be Maxwellian, the conclusions that we have drawn from the simulations remain qualitatively valid for other distributions, as long as one treats the average of the longitudinal kinetic energy spread in the beam frame as the effective longitudinal temperature.

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