An Exact Magnetic-Moment Invariant of Charged-Particle Gyromotion

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For the motion of a charged particle in a uniform, time-dependent axial magnetic field $B(t)\mathbf{e}_z$, it is shown that there is an exact magnetic-moment invariant of the particle dynamics M, to which the adiabatic magnetic-moment invariant $\mu = mv_\perp^2/2B$ is asymptotic when the time scale of the magnetic field variation is much slower than the gyroperiod. The connection between the exact invariant M and the adiabatic invariant μ enables us to characterize in detail the robustness of the adiabatic magnetic-moment invariant μ.

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The magnetic moment of a charged particle gyrating in a magnetic field B is defined as $\mu \equiv mv_\perp^2/2B$, where m is the particle mass and $v_\perp$ is the perpendicular particle speed. As an adiabatic invariant of the particle motion in a magnetic field with slow variations in space and time [1], it is an important concept in plasma physics. Even though it is a classical result in plasma physics [1,2], there is renewed interest in further development [3,4]. In this Letter, we demonstrate that there is an exact magnetic-moment invariant M of a charged particle's gyromotion in a uniform, time-dependent magnetic field $B(t)\mathbf{e}_z$. We also prove that when the time scale of the magnetic field variation is much slower than the gyroperiod, $|(\partial B/\partial t)(1/B)| \ll \omega_c \equiv |qB/mc|$, the magnetic moment μ, as an adiabatic invariant, is asymptotic to the exact invariant M. Furthermore, the connection between the exact invariant M and the adiabatic invariant μ enables us to characterize in detail the robustness of the invariance of the magnetic moment μ, which is an important theoretical underpinning for the magnetic confinement of fusion plasmas.

For present purpose, we consider the nonrelativisitc motion of a charged particle with mass m and charge q in the uniform, time-dependent magnetic field $\mathbf{B} = B(t)\mathbf{e}_z$ inside a long, tightly wound solenoid aligned in the $\mathbf{e}_z$ direction. In this geometry, the vector potential $\mathbf{A}$ that generates $\mathbf{B} = \nabla \times \mathbf{A}$ is

$$\mathbf{A} = A_\theta(r, t)\mathbf{e}_\theta = \tfrac{1}{2}B(t)r\mathbf{e}_\theta, \tag{1}$$

where r is the radial distance from the axis of the solenoid, and $\mathbf{e}_\theta$ is a unit vector in the θ direction in the cylindrical polar coordinates (r, θ, z). The corresponding electric field determined from $\nabla \times \mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t$ is

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial \mathbf{t}} = -\frac{1}{2c}\dot{B}(t)r\mathbf{e}_\theta, \tag{2}$$

where the overdot denotes d/dt. In this field configuration, the z motion of the particle is decoupled from the transverse motion and is described trivially by $\ddot{z} = 0$, corresponding to constant axial velocity with $\dot{z} = v_z = \text{const}$. On the other hand, the Lagrangian of the transverse particle motion is

$$L = \left(\frac{q}{c}\mathbf{A} + m\mathbf{v}_\perp\right)\cdot \mathbf{v}_\perp - \frac{1}{2}m\mathbf{v}_\perp^2 = \frac{q}{c}A_\theta r\dot{\theta} + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2). \tag{3}$$

The transverse canonical momenta associated with Eq. (3) are

$$P_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = \frac{qA_\theta}{c}r + mr^2\dot{\theta}, \tag{4}$$

$$P_r \equiv \frac{\partial L}{\partial \dot{r}} = m\dot{r}. \tag{5}$$

Because $\partial L/\partial\theta = 0$, P_θ is an invariant of the motion. This is of course an elementary result corresponding to the conservation of canonical angular momentum. However, there is another invariant that is associated with the radial dynamics, which is much less transparent. The equation for the dynamics of $r(t)$ is readily shown to be

$$\ddot{r} + \Omega^2(t)r = \frac{P_\theta^2}{m^2r^3}, \tag{6}$$

where $\Omega \equiv qB(t)/2mc$ is one-half of the instantaneous gyrofrequency (also called the Larmor frequency). To construct the expected invariant, we make use of the following result [5].

Theorem 1.—For an arbitrary function $\kappa(t)$ and y_1, y_2 satisfying

$$\ddot{y}_1 + \kappa y_1 = \frac{c_1}{y_1^3}, \tag{7}$$

$$\ddot{y}_2 + \kappa y_2 = \frac{c_2}{y_2^3}, \tag{8}$$

where c_1 and c_2 are real constants, the quantity

$$I = c_1\left(\frac{y_2}{y_1}\right)^2 + c_2\left(\frac{y_1}{y_2}\right)^2 + (y_2\dot{y}_1 - \dot{y}_2y_1)^2 \tag{9}$$

is an invariant with $I = \text{const}$ (independent of t).

Proof.—Calculating $\dot{I} = dI/dt$ from Eqs. (7) and (8) readily gives

$$\begin{aligned}\dot{I} &= 2(y_1\dot{y}_2 - \dot{y}_1 y_2)\left[\frac{c_1 y_2}{y_1^3} + \frac{c_2 y_1}{y_2^3} + (y_1\ddot{y}_2 - \ddot{y}_1 y_2)\right] \\ &= 2(y_1\dot{y}_2 - \dot{y}_1 y_2)[y_2(\ddot{y}_1 + \kappa y_1) - y_1(\ddot{y}_2 + \kappa y_2) \\ &\quad + (y_1\ddot{y}_2 - \ddot{y}_1 y_2)] \\ &= 0.\end{aligned}$$

Even though it is straightforward to prove the theorem directly, the form of the invariant is difficult to guess from elementary considerations. The invariant I in the form of Eq. (9) was first obtained by analyzing the symmetry properties of Eqs. (7) and (8) [5]. Symplectic structure and symmetry properties of the general nonautonomous Hamiltonian systems were studied by Struckmeier and Riedel [6,7]. A constructive proof of Theorem 1 using elementary methods can be given through the following transformation [8]. Letting $y_1(t) = y_2(t)z(t)$, we rewrite Eq. (7) as

$$\ddot{y}_2 z + 2\dot{y}_2\dot{z} + y_2\ddot{z} + \kappa y_2 z = \frac{c_1}{y_2^3 z^3}. \tag{10}$$

If we choose $y_2(t)$ to satisfy Eq. (8), then Eq. (10) becomes

$$\frac{c_2 z}{y_2^3} - \frac{c_1}{y_2^3 z^3} + 2\dot{y}_2\dot{z} + y_2\ddot{z} = 0,$$

which can be integrated once when multiplied by $y_2^3\dot{z}$ to give

$$c_2 z^2 + c_1\frac{1}{z^2} + y_2^4\dot{z}^2 = \text{const.} \tag{11}$$

Equation (11) is identical to Eq. (9). What is demonstrated by the above constructive proof is that there is a close connection (or a symmetry) between the solutions of Eqs. (7) and (8).

There are two ways to interpret the invariant I. First, it can be viewed as an invariant for a two-particle system despite the fact that the dynamics of the two particles is decoupled. Struckmeier and Riedel derived an exact invariant for 3D Hamiltonian systems of N particles confined within a general velocity-dependent potential [9,10]. Another interpretation is that it is an invariant of the single-particle dynamics $y_1(t)$, constructed using a precalculated auxiliary function $y_2(t)$. At first glance, one may suspect the fact that I in the form of Eq. (9) is a valid invariant for the dynamics of $y_1(t)$ described by Eq. (7) because I depends on the solution of $y_2(t)$. However, it is readily demonstrated that I is indeed a valid invariant for $y_1(t)$ by the fact that $y_2(t)$ is completely independent of Eq. (7). It is only necessary to pick a set of arbitrary initial conditions and c_2 to solve for $y_2(t)$ once, and this particular function can then be used to construct invariants for all of the particle dynamics described by Eq. (7) with different initial conditions and constant c_1. The functionality of $y_2(t)$ occurring in I is similar to a special function defined by an ordinary differential equation, such as the Bessel function. It is of course common for an invariant to depend on one or several special functions and their derivatives. Furthermore, the invariant I can be viewed as a generalized Courant-Snyder invariant [11–13], which is a basic result regarding charged-particle dynamics in a transverse focusing lattice in particle accelerators. The Courant-Snyder invariant was rediscovered by Lewis [14], and is sometimes referred as the Lewis invariant by different authors.

From Theorem 1, the transverse motion of the charged particle has an exact invariant I_α given by

$$I_\alpha = \frac{P_\theta^2}{m^2}\left(\frac{w_\alpha^2}{r^2}\right) + \alpha\left(\frac{r^2}{w_\alpha^2}\right) + (\dot{r}w_\alpha - \dot{w}_\alpha r)^2, \tag{12}$$

where α is an arbitrary real constant and $w_\alpha(t)$ is any function of time satisfying

$$\ddot{w}_\alpha + \Omega^2(t)w_\alpha = \frac{\alpha}{w_\alpha^3}. \tag{13}$$

The invariant I_α in Eq. (12) will be recognized as a linear combination of the constants of the motion, A_x^2 and A_y^2, used in accelerator physics [12,13], where A_x and A_y are the constant amplitude scale factors for the transverse particle orbits in a time-varying magnetic field $B(t)\mathbf{e}_z$, and πA_x^2 and πA_y^2 are the corresponding (conserved) phase-space areas.

For present purpose, we assume $qB(t) > 0$. Of course, any function of P_θ and I_α is also an invariant of the single-particle motion. Of particular interest is the invariant M defined by

$$\begin{aligned}M &\equiv \frac{q}{4c}\left(I_1 - \frac{2P_\theta}{m}\right) \\ &= \frac{qw_1^2}{4c}\left\{\left(\dot{r} - r\frac{\dot{w}_1}{w_1}\right)^2 + r^2\left[\left(\frac{qB}{2mc} + \dot{\theta}\right) - \frac{1}{w_1^2}\right]^2\right\},\end{aligned} \tag{14}$$

where I_1 is equal to I_α in Eq. (12) with $\alpha = 1$, and use has been made of Eq. (4). In the subsequent analysis, we refer to M as the exact magnetic-moment invariant. [For the case of $qB(t) < 0$, the definition of M is identical to Eq. (14) with I_1 replaced by $-I_1$, and the Ω in Eqs. (17)–(21) is replaced by $-\Omega$.]

When the magnetic field variation is slow compared with the gyroperiod, i.e.,

$$B = B(\varepsilon t), \tag{15}$$

where $\varepsilon \ll 1$, the magnetic moment $\mu = mv_\perp^2/2B(t) = m(\dot{r}^2 + r^2\dot{\theta}^2)/2B(t)$ is a well-known adiabatic invariant. We now prove that μ is asymptotic to the exact invariant M for small ε. Let $\Omega = \Omega(\varepsilon t)$ and $T = \varepsilon t$. Then Eq. (13) for $\alpha = 1$ becomes

$$\varepsilon^2 \frac{d^2 w_1}{dT^2} + \Omega^2(T) w_1 = \frac{1}{w_1^3}. \tag{16}$$

Expressing $w_1 = u_0 + u_1\varepsilon + u_2\varepsilon^2 + \cdots$, it is straightforward to show from Eq. (16) that

$$u_0 = \frac{1}{\Omega^{1/2}}, \tag{17}$$

$$u_1 = 0, \tag{18}$$

$$u_2 = -\frac{3}{4\Omega^2}\frac{d^2}{dT^2}(\Omega^{1/2}), \qquad \ldots, \tag{19}$$

where $\Omega(T) \equiv qB(T)/2mc > 0$ is assumed. Therefore,

$$w_1 = \frac{1}{\Omega^{1/2}} + O(\varepsilon^2), \tag{20}$$

$$\dot{w}_1 = O(\varepsilon). \tag{21}$$

Substituting w_1 and $\dot{w}_1$ into Eq. (14), we readily obtain

$$M = \frac{q}{4c\Omega}[\dot{r}^2 + r^2\dot{\theta}^2] + O(\varepsilon) = \mu + O(\varepsilon), \tag{22}$$

which shows the relationship between the adiabatic invariant μ and the exact invariant M for $\varepsilon \ll 1$. Equation (22) also enables us to establish two very important properties of the adiabatic invariant μ. These two properties allow us to quantify the exact meaning of the adjective "adiabatic." The first property of μ pertains to the change of μ at any time t relative to its value at $t = 0$. From Eq. (22),

$$\mu(t) = M + O(\varepsilon), \tag{23}$$

$$\mu(0) = M + O(\varepsilon). \tag{24}$$

Because M is an exact invariant, it follows that

$$\Delta\mu(t) \equiv \mu(t) - \mu(0) = O(\varepsilon) \tag{25}$$

for all t. In other words, the change of μ is always small for all t. This is a powerful statement for two reasons. First, from the definition of μ and Eq. (6),

$$\frac{d\mu}{dt} = -\frac{\dot{B}}{B}\left(\frac{mv_\perp^2}{2} + \frac{P_\theta\Omega}{2} - \frac{mr^2\Omega^2}{4}\right) = O(\varepsilon). \tag{26}$$

In general, we would expect $\Delta\mu \sim O(\varepsilon^{1-n})$ for $t \sim O(\varepsilon^{-n})$, if μ did not have the extra dynamical properties described above. Second, in order to qualify to be called an adiabatic invariant, it is only required that the change of the quantity be $O(\varepsilon)$ for $0 \le t \le O(1/\varepsilon)$ [15]. What we have proved is a stronger result that $\Delta\mu = O(\varepsilon)$ for all t

The second important property of μ concerns the difference between the final state and the initial state when Ω evolves from an initial constant value to a final constant value. This property can be stated as follows. If

$$\Omega > \Omega_0 > 0, \qquad \lim_{T\to+\infty}\Omega(T) = \Omega_+, \qquad \lim_{T\to-\infty}\Omega(T) = \Omega_-, \tag{27}$$

and

$$\lim_{T\to\pm\infty}\frac{d^i\Omega}{dT^i} \text{ exists for } i \ge 1, \tag{28}$$

then for any integer n,

$$\mu(+\infty) - \mu(-\infty) = o(\varepsilon^n). \tag{29}$$

This type of characterization of an adiabatic invariant was first adopted by Kulsrud [16]. To prove Eq. (29) under the conditions in Eqs. (27) and (28), we carry out a perturbative analysis of Eq. (16) to order n for any integer n. Let $w_1 = \sum_n u_n\varepsilon^n$ and $S = \sum_n S_n\varepsilon^n = 1/w_1^3$. Obviously, $S_0 = 1/u_0^3$ and $S_1 = 0$. For $n \ge 2$,

$$u_n\Omega^2 = S_n - \ddot{u}_{n-2}, \qquad S_n = -\frac{3S_0}{u_0}u_n - \frac{1}{u_0^3}\sum_{i,j,k=0}^{n-1} S_{n-1-(i+j+k)}u_iu_ju_k, \tag{30}$$

where $S_l = 0$ for $l < 0$. From this iteration relation and the fact that u_1 and u_2 are homogeneous polynomials in terms of $\dot{\Omega}$ and $\ddot{\Omega}$, we can deduce that u_n is a homogeneous polynomial in terms of $d^i\Omega/dT^i$ $(i = 1, \ldots, n)$. Furthermore, because $\lim_{T\to\pm\infty} d^i\Omega/dT^i$ exists for $i \ge 0$, it follows that

$$\lim_{T\to\pm\infty}\frac{d^i\Omega}{dT^i} = 0, \qquad (i \ge 1). \tag{31}$$

There exists a T_n such that when $T > T_n$ and $T < -T_n$,

$$\left|\frac{d^i\Omega}{dT^i}\right| < \varepsilon^{n+1}, \qquad (i = 1, \ldots, n). \tag{32}$$

Therefore, for $T > T_n$ and $T < -T_n$, we obtain

$$w_1 = \frac{1}{\Omega^{1/2}} + o(\varepsilon^n), \tag{33}$$

$$w_1' = o(\varepsilon^n), \tag{34}$$

and for $t > T_n/\varepsilon$ and $t < -T_n/\varepsilon$,

$$M = \mu(t) + o(\varepsilon^n). \tag{35}$$

In Eq. (34), $w_1' = dw_1/dT$. Consequently,

$$\mu(t) - \mu(-t) = o(\varepsilon^n) \tag{36}$$

for $t > T_n/\varepsilon$, and we have proved the result stated in Eq. (29).

As a final point, it should also be emphasized that the existence of the exact invariants I_1 and P_θ represents powerful constraint conditions that can be used to determine exact expressions for the transverse orbits $r(t)$ for general initial conditions at $t = 0$. To illustrate this point, we denote the particular solution to Eq. (13) for $\alpha = 1$ by $w_1(t)$. For a prescribed functional form for $\Omega(t)$, and specified initial conditions $w_1|_{t=0}$ and $dw_1/dt|_{t=0}$, the solution for $w_1(t)$ can be determined numerically from Eq. (13). We now introduce the stretched time variable $\tau(t)$ and the scaled radial coordinate $R(\tau)$ defined by

$$\tau = \int_0^t \frac{dt}{w_1^2(t)}, \qquad R = \frac{r}{w_1}. \tag{37}$$

For $\alpha = 1$, Eq. (9) can then be expressed as

$$\left(\frac{d}{d\tau}R^2\right)^2 + 4\left(R^2 - \frac{1}{2}I_1\right)^2 = \left(I_1^2 - \frac{4P_\theta^2}{m^2}\right). \tag{38}$$

From Eqs. (37) and (38), $R^2(\tau) - I_1/2$ exhibits simple harmonic motion proportional to $\cos(2\tau)$ and $\sin(2\tau)$, and the exact solution for $r^2(t)$ can be expressed as

$$r^2(t) = w_1^2(t)\left[\frac{1}{2}I_1 + \frac{1}{2}\left(I_1^2 - \frac{4P_\theta^2}{m^2}\right)^{1/2} \times \cos\left(2\int_0^t \frac{dt}{w_1^2(t)} + \phi_0\right)\right], \tag{39}$$

where ϕ_0 is a constant phase factor.

In conclusion, for the case of a uniform, time-dependent magnetic field $B(t)\mathbf{e}_z$, we have demonstrated that there is an exact invariant I_α associated with the transverse particle dynamics. An exact magnetic-moment invariant M was constructed, to which the adiabatic invariant $\mu = mv_\perp^2/2B$ is asymptotic when the time scale of the gyromotion is fast in comparison with the time scale for variation in $B(t)$. The relation between the exact invariant M and the adiabatic invariant μ has enabled us to quantify several important properties regarding the robustness of the adiabatic invariant μ. Besides its importance to the theory of magnetic confinement, there are other interesting applications of the theory developed here. One example is the concept of subharmonic heating and cooling. It is well known that charged particles in a magnetic field can be heated or cooled by ramping-up or ramping-down the magnetic field. However, this magnetic pumping effect offers a limited heating or cooling capability, because the field cannot be ramped-up or -down indefinitely. It is ideal if particles can be heated or cooled in a periodically varying magnetic field. But, the approximate invariance of the magnetic moment indicates that to leading order, a particle's kinetic energy is conserved in one full cycle of the magnetic field. The exact magnetic-moment invariant M can be used to calculate the next-order kinetic energy variation, which can be increasingly significant with increasing pumping frequency. Such a magnetic heating or cooling technique may prove valuable in plasma physics and accelerator physics applications. The case considered here does not include spatial inhomogeneities in the magnetic field. The general case with space-time variations in $\mathbf{B}(\mathbf{x}, t)$ will be the subject of a subsequent investigation. Here, we make the following conjecture: under the most general conditions for the magnetic moment $\mu = mv_\perp^2/2B(\mathbf{x}, t)$ to be an adiabatic invariant, for most particles there exist exact invariants of the transverse particle dynamics, to which the magnetic moment is asymptotic. Such invariants correspond to the invariant tori of the Kolmogorov-Arnold-Moser theorem when the deviation from an integrable system is small enough. The Kolmogorov-Arnold-Moser theorem guarantees the existence of these surfaces by proving the convergence of the perturbation series for the tori. The methods adopted in this Letter are a direct construction of the invariant tori using independent special functions determined from several differential equations describing the symmetry properties of the perturbed system.

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