# Wall-Impedance-Driven Collective Instability in Intense Charged Particle Beams

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## Abstract

The linearized Vlasov-Maxwell equations are used to investigate detailed properties of the wall-impedance-driven instability for a long charge bunch (bunch length $\ell_b \gg$ bunch radius $r_b$) propagating through a cylindrical pipe with radius $r_w$ and wall impedance $\tilde{Z}(\omega)$. The stability analysis is carried out for perturbations with azimuthal mode number $\ell \geq 1$ about a cylindrical Kapchinskij-Vladimirskij (KV) beam equilibrium with flattop density profile in the smooth-focusing approximation. Detailed stability properties are determined for dipole-mode perturbations ($\ell = 1$) assuming negligibly small axial momentum spread of the beam particles. The stability analysis is valid for general value of the normalized beam intensity $s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2$ in the interval $0 < s_b < 1$, where $\hat{\omega}_{pb} = (4\pi\hat{n}_b e_b^2/\gamma_b m_b)^{1/2}$ is the relativistic plasma frequency and $\omega_{\beta\perp}$ is the applied focusing frequency.

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## I. INTRODUCTION

High energy ion accelerators, transport systems and storage rings[1–8] have a wide range of applications ranging from basic research in high energy and nuclear physics, to applications such as spallation neutron sources, heavy ion fusion, and nuclear waste transmutation. Charged particle beams are subject to various collective instabilities that can deteriorate the beam quality. Of particular importance at the high beam currents and charge densities of practical interest are the effects of the intense self-fields produced by the beam space charge and current on determining detailed equilibrium, stability, and transport properties. In general, a complete description of collective processes in intense charged particle beams is provided by the nonlinear Vlasov-Maxwell equations[1] for the self-consistent evolution of the beam distribution function, $f_b( x, \mathbf{p}, \mathbf{t})$, and the electric and magnetic fields, $\mathbf{E}(\mathbf{x}, \mathbf{t})$ and $\mathbf{B}(\mathbf{x}, \mathbf{t})$. While considerable progress has been made in analytical and numerical simulation studies of intense beam propagation[9–40], the effects of finite geometry and intense self-fields often make it difficult to obtain detailed predictions of beam equilibrium, stability, and transport properties based on the Vlasov-Maxwell equations. Nonetheless, often with the aid of numerical simulations, there has been considerable recent analytical progress in applying the Vlasov-Maxwell equations to investigate the detailed equilibrium and stability properties of intense charged particle beams. These investigations include a wide variety of diverse applications ranging from the Harris-like instability driven by large temperature anisotropy with $T_{\perp b} \gg T_{\| b}$[37], to the dipole-mode two-stream instability for an intense ion beam propagating through background electrons[38], to the resistive hose instability[39] and the sausage and hollowing instabilities[40] for intense beam propagation through background plasma, to the development of a nonlinear stability theorem[22, 23] in the smooth-focusing approximation. Building on these advances[1, 37–40], in the present analysis we reexamine the classical wall-impedance-driven instability[41–45], also called the resistive-wall instability, making use of the linearized Vlasov-Maxwell equations[1] for perturbations about a Kapchinskij-Vladimirskij (KV) beam equilibrium $f_b^0(\mathbf{x}, \mathbf{p})$[9–11] with flattop density profile.

To briefly summarize, the present analysis assumes a very long charge bunch (bunch length $\ell_b \gg$ bunch radius $r_b$) with directed axial kinetic energy $(\gamma_b - 1)m_b c^2$ propagating in the $z$-direction through a cylindrical pipe with constant radius $r_w$ and (complex) wall impedance $\tilde{Z}(\omega)$[2]. The analysis is carried out in the smooth-focusing approximation, where the applied transverse focusing force is modeled by $\mathbf{F}_{foc} = -\gamma_b m_b \omega_{\beta\perp}^2 \mathbf{x}_\perp$. Here, $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor, $V_b = \beta_b c$ is the directed axial velocity of the charge bunch, $m_b$ is the particle rest mass, $\omega_{\beta\perp} = const.$ is the applied focusing frequency, and $\mathbf{x}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ is the transverse displacement of a beam particle from the cylinder axis. Denoting the number density of beam particles by $\hat{n}_b$ and the particle charge by $e_b$, it is convenient to introduce the relativistic plasma frequency $\hat{\omega}_{pb}$ defined by $\hat{\omega}_{pb} = (4\pi\hat{n}_b e_b^2/\gamma_b m_b)^{1/2}$ and the normalized (dimensionless) beam intensity $s_b$ defined by $s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2$[1].

An important feature of the present analysis of the linearized Vlasov-Maxwell equations is that it is carried out for arbitrary value of the normalized beam intensity in the interval $0 < s_b < 1$, assuming perturbations about a KV beam equilibrium with flattop density profile $n_b^0(\mathbf{x}) = \int d^3p f_b^0(\mathbf{x}, \mathbf{p})$. Illustrative parameters for intense beam systems are shown in Table 1 for the Tevatron[46], for coasting beam experiments in the Proton Storage Ring[47, 48], and for the space-charge-dominated beams envisioned for heavy ion fusion[8]. Note from Table 1 that the normalized beam intensity $s_b$ ranges from the very small value $s_b = 1.36 \times 10^{-4}$ in the Tevatron, where the particles are highly relativistic, to the intermediate value $s_b = 0.08$ in the low-energy, moderate-intensity Proton Storage Ring experiment, to $s_b \simeq 0.98$ in the low-emittance, space-charge-dominated beams for heavy ion fusion. In any case, the present kinetic analysis of the wall-impedance-driven instability is carried out for arbitrary value of normalized-beam intensity $s_b$ in the interval $0 < s_b < 1$, and (in principle) can be applied to the diverse range of high-intensity beam systems in Table 1. Finally, the present analysis considers the case where the axial momentum spread is negligibly small, and the corresponding Landau damping[1] by parallel kinetic effects is absent. (This gives a larger estimate of the instability growth rate than would be obtained with finite axial momentum spread.) Furthermore, the functional form of the wall impedance $\tilde{Z}(\omega)$ is not specified, although the case of small impedance ($|\tilde{Z}| \ll 1$) is considered when analyzing the kinetic dispersion relation in Secs. III and IV.

The organization of this paper is the following. The theoretical model and assumptions are summarized in Sec. I. In Secs. II and III, the detailed kinetic stability analysis is carried out for perturbations about a KV beam equilibrium with flattop density profile, leading to the kinetic dispersion relation (58), valid for arbitrary multipole perturbations with azimuthal mode number $\ell \geq 1$ about an axisymmetric beam equilibrium. Finally, in Sec. IV detailed properties of the wall-impedance-driven instability are calculated for dipole-mode perturbations ($\ell = 1$) and general values of the normalized beam intensity $s_b$ in the interval $0 < s_b < 1$.

TABLE I: Illustrative parameters for intense beam systems

| | Tevatron | Heavy Ion Fusion Driver | PSR |
|---|---|---|---|
| Ion | p | $Cs^+$ | p |
| Mass number (A) | 1 | 133 | 1 |
| Kinetic energy $(\gamma_b - 1)m_b c^2$ (GeV) | 150 | 2.5 | 0.8 |
| Relativistic $\gamma_b$ | 160 | 1.02 | 1.85 |
| Wall radius $r_w$ (cm) | 2.5 | 9 | 5 |
| Beam radius $r_b$(cm) | 0.44 | 4.24 | 2.63 |
| Bunch length $l_b$ (cm) | 37 | 1000 | 6000 |
| $l_b/r_b$ | 84.7 | 236 | 2281 |
| Focusing frequency $\omega_{\beta\perp}$ ($s^{-1}$) | $6.17 \times 10^6$ | $1.9 \times 10^7$ | $4.0 \times 10^7$ |
| Beam density $\hat{n}_b$ ($cm^{-3}$) | $2.4 \times 10^{10}$ | $5.6 \times 10^{10}$ | $9.4 \times 10^8$ |
| Plasma frequency $\hat{\omega}_{pb}$ ($s^{-1}$) | $1.6 \times 10^7$ | $2.7 \times 10^7$ | $3.0 \times 10^7$ |
| Emittance $\varepsilon_N$ (mm-mrad) | $20\pi$ | $7.7\pi$ | $45\pi$ |
| Normalized intensity $s_b$ | $1.36 \times 10^{-4}$ | 0.98 | 0.08 |

## II. THEORETICAL MODEL AND ASSUMPTIONS

The present analysis considers a very long charge bunch with characteristic axial length $\ell_b$ and radius $r_b$ satisfying $\ell_b \gg r_b$. The charge bunch is made up of particles with charge $e_b$ and rest mass $m_b$ propagating in the $z$-direction with directed axial kinetic energy $(\gamma_b - 1)m_b c^2$, where $\gamma_b = (1 - \beta_b^2)^{1/2}$ is the relativistic mass factor, $V_b = \beta_b c$ is the average axial velocity, and $c$ is the speed of light *in vacuo*. The charge bunch propagates through a cylindrical, conducting pipe with wall radius $r_w$, and the applied transverse focusing force on a beam particle is modeled in the smooth focusing approximation by

$$\mathbf{F}_{foc} = -\gamma_b m_b \omega_{\beta\perp}^2 \mathbf{x}_\perp \ , \tag{1}$$

where $\omega_{\beta\perp} = const.$ is the applied focusing frequency, and $\mathbf{x}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ is the transverse displacement of a beam particle from the cylinder axis at $r = 0$. Furthermore, the particle motion in the beam frame is treated in the paraxial approximation with $p_x^2$, $p_y^2$, $(p_z - \gamma_b m_b \beta_b c)^2 \ll \gamma_b^2 m_b^2 \beta_b^2 c^2$.

To describe stability properties of the charge bunch, we make use of a kinetic description based on the Vlasov-Maxwell equations, which describe the self-consistent nonlinear evolution of the distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$ and the self-generated electric and magnetic fields, $\mathbf{E}^s(\mathbf{x}, t)$ and $\mathbf{B}^s(\mathbf{x}, t)$, in the six-dimensional phase space $(\mathbf{x}, \mathbf{p})$. For simplicity, the present analysis considers small-amplitude perturbations about the axisymmetric $(\partial/\partial\theta = 0)$, axially uniform $(\partial/\partial z = 0)$, quasi-steady-state $(\partial/\partial t = 0)$ equilibrium distribution function [38]

$$f_b^0(r, \mathbf{p}_\perp) = \frac{\hat{n}_b}{2\pi\gamma_b m_b}\delta(H_\perp - \hat{T}_{\perp b})\delta(p_z - \gamma_b m_b \beta_b c) \ . \tag{2}$$

In Eq. (2), $\hat{n}_b$ and $\hat{T}_{\perp b}$ are positive constants, and $H_\perp$ is the transverse Hamiltonian defined by

$$H_\perp = \frac{1}{2\gamma_b m_b} p_\perp^2 + \frac{1}{2}\gamma_b m_b \omega_{\beta\perp}^2 r^2 + e_b[\phi^0(r) - \beta_b A_z^0(r)] \ , \tag{3}$$

where $r = (x^2 + y^2)^{1/2}$ is the radial distance from the cylinder axis, and $p_\perp = (p_x^2 + p_y^2)^{1/2}$ is the transverse momentum. In Eq. (3), the equilibrium self-field potentials, $\phi^0(r)$ and $A_z^0(r)$, are determined self-consistently in terms of $f_b^0(r, \mathbf{p})$ from the steady-state Maxwell equations. Because of the delta-function dependence on $p_z$, note that the choice of distribution function in Eq. (2) is *cold* in the axial direction. An attractive feature of the choice of $f_b^0(r, \mathbf{p})$ in Eq. (2) is that the corresponding equilibrium number density, $n_b^0(r) = \int d^3p f_b^0(r, \mathbf{p})$, has the flattop profile[38]

$$n_b^0(r) = \begin{cases} \hat{n}_b = const., & 0 \le r < r_b \ , \\ 0 \ , & r_b < r \le r_w \ . \end{cases} \tag{4}$$

Here, $\hat{n}_b = const.$ is the number density of beam particles, and the edge radius $r_b$ is determined self-consistently from

$$\frac{2\hat{T}_{\perp b}}{\gamma_b m_b} = \left(\omega_{\beta\perp}^2 - \frac{1}{2\gamma_b^2}\hat{\omega}_{pb}^2\right) r_b^2 \equiv \nu_b^2 r_b^2 \tag{5}$$

where $\hat{\omega}_{pb}^2 = 4\pi\hat{n}_b e_b^2/\gamma_b m_b$ is the relativistic plasma frequency-squared. Here, we have introduced the quantity $\nu_b^2$ defined by

$$\nu_b^2 = \omega_{\beta\perp}^2 - \frac{1}{2\gamma_b^2}\hat{\omega}_{pb}^2 = \omega_{\beta\perp}^2(1 - s_b) \ , \tag{6}$$

where

$$s_b = \frac{\hat{\omega}_{pb}^2}{2\gamma_b^2 \omega_{\beta\perp}^2} \tag{7}$$

is a convenient dimensionless measure of the normalized beam intensity. Note from Eq. (6) that $\nu_b = \omega_{\beta\perp}(1-s_b)^{1/2}$ corresponds to the (depressed) betatron frequency for transverse particle oscillations in the equilibrium field configuration. For parameters typical of the Tevatron[46], $s_b \ll 1$ and $\nu_b \simeq \omega_{\beta\perp}$, corresponding to very weak equilibrium self fields. For parameters typical of heavy ion fusion applications[8], however, $s_b$ is in the range $0.9 < s_b < 1$, corresponding to very large tune depressions. On the other hand, for accelerators used in nuclear physics applications[47, 48], such as the Proton Storage Ring (PSR) facility and the Spallation Neutron Source (SNS), the intensity parameter $s_b$ is in the intermediate range, $0.05 < s_b < 0.2$.

An important goal of the present analysis is to develop a theoretical model that determines the effects of finite wall impedance and is valid over the entire range of normalized beam intensity, $0 < s_b < 1$. To this end, we express $f_b(\mathbf{x},\mathbf{p},t) = f_b^0(r,\mathbf{p}) + \delta f_b(\mathbf{x},\mathbf{p},t)$, and make use of the linearized Vlasov-Maxwell equations[1, 38] to determine the self-consistent evolution of $\delta f_b(\mathbf{x},\mathbf{p},t)$, $\delta\mathbf{E}^s(\mathbf{x},t)$ and $\delta\mathbf{B}^s(\mathbf{x},t)$ for small-amplitude perturbations. For perturbations about the equilibrium distribution function $f_b^0(r,\mathbf{p})$ in Eq. (2), the linearized Vlasov equation for $\delta f_b(\mathbf{x},\mathbf{p},t)$ can be expressed as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \gamma_b m_b \nu_b^2 \mathbf{x}_\perp \cdot \frac{\partial}{\partial \mathbf{p}}\right)\delta f_b = -\frac{1}{\gamma_b m_b}\delta\mathbf{F}_\perp \cdot \mathbf{p}_\perp \frac{\partial f_b^0}{\partial H_\perp} - \delta F_z \frac{\partial}{\partial p_z} f_b^0 \ , \tag{8}$$

where $\delta\mathbf{F}_\perp = e_b(\delta\mathbf{E}^s + \mathbf{v}\times\delta\mathbf{B}^s/c)_\perp$ and $\delta F_z = e_b(\delta\mathbf{E}^s + \mathbf{v}\times\delta\mathbf{B}^s/c)_z$ are the perturbed transverse and longitudinal forces. We further express $\delta\mathbf{E}^s = -\nabla\delta\phi - c^{-1}\partial\delta\mathbf{A}/\partial t$ and $\delta\mathbf{B}^s = \nabla\times\delta\mathbf{A}$, and make use of the Lorentz gauge condition, $\nabla\cdot\delta\mathbf{A} = -c^{-1}\partial\delta\phi/\partial t$, to relate $\delta\mathbf{A}$ and $\delta\phi$. The linearized Maxwell equations for $\delta\phi(\mathbf{x},t)$ and $\delta\mathbf{A}(\mathbf{x},t)$ can then be expressed as

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\delta\phi = -4\pi e_b \int d^3p\,\delta f_b \ , \tag{9}$$

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\delta\mathbf{A} = -\frac{4\pi e_b}{c} \int d^3p\,\mathbf{v}\,\delta f_b \ . \tag{10}$$

Here, $\mathbf{v} = \mathbf{p}/\gamma m_b$ is the particle velocity, $\gamma = (1+\mathbf{p}^2/m_b^2c^2)^{1/2}$ is the kinematic mass factor, and $\nabla_\perp^2 = (1/r)(\partial/\partial r)(r\partial/\partial r) + r^{-2}\partial^2/\partial\theta^2$ is the perpendicular Laplacian in cylindrical polar coordinates $(r,\theta,z)$. Note that Eqs. (9) and (10) determine the perturbed self-field potentials, $\delta\phi(\mathbf{x},t)$ and $\delta\mathbf{A}(\mathbf{x},t)$, in terms of the perturbed charge and current densities, $\delta\rho_b(\mathbf{x},t) = e_b\int d^3p\delta f_b(\mathbf{x},\mathbf{p},t)$ and $\delta\mathbf{J}_b(\mathbf{x},t) = e_b\int d^3p\mathbf{v}\delta f_b(\mathbf{x},\mathbf{p},t)$, where $\delta f_b(\mathbf{x},\mathbf{p},t)$ is determined self-consistently from Eq. (8).

In Sec. 3, Eqs. (8)–(10) will be analyzed for perturbations of the form

$$\delta\psi(\mathbf{x},t) = \delta\psi^\ell(r)\exp(i\ell\theta + ik_z z - i\omega t) \; , \tag{11}$$

where $\ell = 1, 2, \ldots$ is the azimuthal mode number of the perturbation, $k_z$ is the axial wavenumber, and $\omega$ is the oscillation frequency. For perturbations with real $\omega$ and $Imk_z < 0$, the perturbation is growing spatially as a function of $z$. On the other hand, for perturbations with real $k_z$ and $Im\omega > 0$, the perturbation is growing temporally as a function of $t$. For present purposes, we consider perturbations with sufficiently low frequency and long axial wavelength that

$$\frac{|\omega|r_b}{c} \ll 1 \quad \text{and} \quad |k_z|r_b \ll 1 \; . \tag{12}$$

Equations (8)–(10) can be simplified within the context of the inequalities in Eq. (12). For example, making use of the Lorentz gauge condition, $\nabla_\perp \cdot \delta\mathbf{A}_\perp + (\partial/\partial z)\delta A_z = -c^{-1}\partial\delta\phi/\partial t$, it can be shown that $|\delta\mathbf{A}_\perp| \sim r_b|k_z\delta A_z|$ or $r_b|(\omega/c)\delta\phi|$ over the transverse dimensions of the beam. Without presenting algebraic details[1], it therefore follows within the context of Eq. (12) that the $\delta\mathbf{A}_\perp$ contributions in Eqs. (8)–(10) can be neglected and that the perturbed transverse force $\delta\mathbf{F}_\perp$ can be approximated by

$$\delta\mathbf{F}_\perp = -e_b\nabla_\perp\left(\delta\phi - \frac{1}{c}v_z\delta A_z\right) \; . \tag{13}$$

Similarly, for the low-frequency, long-wavelength perturbations consistent with Eq. (12), it can be shown that the perturbed longitudinal force term (proportional to $\delta F_z$) in Eq. (8) can be neglected[38]. Moreover, because the axial momentum spread is negligibly small for the distribution function in Eq. (2), we approximate $\int d^3pv_z\delta f_b = \beta_b c\int d^3p\delta f_b$ in Eq. (10).

In summary, making use of the approximations outlined in the previous paragraph, the linearized Vlasov-Maxwell equations (8)–(10) can be approximated by

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}} - \gamma_b m_b\nu_b^2\mathbf{x}_\perp\cdot\frac{\partial}{\partial\mathbf{p}}\right)\delta f_b = \frac{e_b}{\gamma_b m_b}\mathbf{p}_\perp\cdot\nabla_\perp\left(\delta\phi - \frac{1}{c}v_z\delta A_z\right)\frac{\partial f_b^0}{\partial H_\perp} \; , \tag{14}$$

where $\delta\phi$ and $\delta A_z$ are determined from

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\delta\phi = -4\pi e_b\delta n_b \; , \tag{15}$$

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\delta A_z = -4\pi e_b \beta_b \delta n_b \ . \tag{16}$$

Here, $V_b = \beta_b c$ is the average axial velocity, and $\delta n_b(\mathbf{x}, t)$ is the perturbed number density of beam particles defined in terms of $\delta f_b(\mathbf{x}, \mathbf{p}, t)$ by

$$\delta n_b = \int d^3p \delta f_b \ . \tag{17}$$

Equations (14)–(17) represent the final form of the linearized Vlasov-Maxwell equations used in the stability analysis in Sec. III, carried out for perturbations about the choice of equilibrium distribution function $f_b^0(r, \mathbf{p})$ in Eq. (2) with flattop density profile in Eq. (4).

Equations (14)–(16) are to be solved in the beam interior ($0 \leq r < r_b$) and in the vacuum region ($r_b < r \leq r_w$) outside the beam, enforcing the appropriate boundary conditions at the conducting wall located at radius $r = r_w$. For present purposes, we assume that the wall impedance is described by a complex scalar function, $\tilde{Z}(\omega) = \tilde{Z}_r + i\tilde{Z}_i$, where $\omega$ is the oscillation frequency in Eq. (11), and that the boundary condition on the perturbed tangential electric $E_t$ and magnetic $H_t$ fields at $r = r_w^- \equiv [r_w(1-\epsilon)]_{\epsilon \to 0^+}$ can be expressed as[49]

$$[\delta \mathbf{E}_t]_{r_w^-} = \tilde{Z}(\omega)\hat{\mathbf{n}} \times [\delta \mathbf{B}_t]_{r_w^-} \ . \tag{18}$$

Here, $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_r$ is a unit vector pointing outward from the cylindrical conducting wall surface. In what follows we assume that the metal wall is almost perfectly conducting, implying that $|\tilde{Z}(\omega)| \ll 1$. Assuming that perturbed quantities vary according to Eq. (11), and making use of $(\nabla \times \delta \mathbf{B})_r = c^{-1}\partial \delta E_r/\partial t$ in the vacuum region, the boundary conditions in Eq. (18) can be expressed as

$$\begin{aligned} &[\delta E_z^\ell]_{r_w^-} = -\tilde{Z}[\delta B_\theta^\ell]_{r_w^-} \\ &[\delta E_\theta^\ell]_{r_w^-} = \tilde{Z}[\delta B_z^\ell]_{r_w^-} = \tilde{Z}\left[\frac{k_z r}{\ell}\delta B_\theta^\ell + \frac{\omega r}{\ell c}\delta E_r^\ell\right]_{r_w^-} \ . \end{aligned} \tag{19}$$

Neglecting contributions involving $\delta \mathbf{A}_\perp$ (which can be done under the assumption that $|\tilde{Z}| \ll 1$), we approximate $\delta B_\theta^\ell = -(\partial/\partial r)\delta A_z^\ell$, $\delta E_r^\ell = -(\partial/\partial r)\delta\phi$, $\delta E_\theta^\ell = -(i\ell/r)\delta\phi^\ell$, and $\delta E_z^\ell = -ik_z\delta\phi^\ell + (i\omega/c)\delta A_z^\ell$ in Eq. (19). The boundary conditions in Eq. (19) then reduce to

$$\begin{aligned} &k_z[\delta\phi^\ell]_{r_w^-} - \frac{\omega}{c}[\delta A_z^\ell]_{r_w^-} = i\tilde{Z}\left[\frac{\partial}{\partial r}\delta A_z^\ell\right]_{r_w^-} , \\ &\frac{\ell}{r_w}[\delta\phi^\ell]_{r_w^-} = -i\tilde{Z}\left\{\frac{\omega}{c}\left[\frac{\partial}{\partial r}\delta\phi^\ell\right]_{r_w^-} + k_z\left[\frac{\partial}{\partial r}\delta A_z^\ell\right]_{r_w^-}\right\} \ . \end{aligned} \tag{20}$$

Equation (20) expresses the boundary conditions at the conducting wall in terms of the impedance $\tilde{Z}(\omega)$ and the perturbed potentials, $\delta\phi$ and $\delta A_z$. In the limit of zero impedance, $\tilde{Z} \to 0$, note that Eq. (20) reduces to $[\delta\phi^\ell]_{r_w^-} = 0 = [\delta A_z^\ell]_{r_w^-}$, corresponding to the boundary conditions expected for a perfectly conducting, cylindrical wall. Depending on the frequency regime, there are several models of wall impedance $\tilde{Z}(\omega)$ that can be used in the boundary conditions in Eq. (20). These range from impedance functions that depend on the wall structure and smoothness[2, 42, 43], to impedance functions that depend on the electrical conductivity of the wall[49]. For example, a common expression for $\tilde{Z}(\omega)$ for a smooth-bore, cylindrical conducting wall is given by[49]

$$\tilde{Z}(\omega) = \left(\frac{\omega}{8\pi\sigma}\right)^{1/2} (1-i) \ , \tag{21}$$

where $\sigma$ is the electrical conductivity of the wall.

In concluding this section, we reiterate that the inequalities $|\omega| r_b/c \ll 1$ and $|k_z| r_b/c \ll 1$ in Eq. (12) have been used to simplify the perturbed force $\delta\mathbf{F}$ in the beam interior ($0 \leq r < r_b$) in the linearized Vlasov equation (14). Insofar as the wall radius $r_w$ is not too far removed from the beam radius $r_b$ ($r_w/r_b \sim 2-3$, say), then $|k_z| r_w/c \ll 1$ and $|\omega| r_b/c \ll 1$ are also good approximations in solving the Maxwell equations (15) and (16) in the vacuum region ($r_b < r \leq r_w$), and the terms proportional to $\partial^2/\partial z^2 - c^{-2}\partial^2/\partial t^2$ can be neglected in Eqs. (15) and (16). This is typically encountered in heavy ion fusion applications[8], and in some accelerators for nuclear physics applications such as the Proton Storage Ring (PSR) facility[36, 47]. In the general case, however, making use of Eq. (11), the solutions to Eqs. (15) and (16) for $\delta\phi^\ell(r)$ and $\delta A_z^\ell(r)$ in the vacuum region are linear combinations of $I_\ell(\kappa r)$ and $K_\ell(\kappa r)$, where $\kappa(k_z,\omega)$ is defined by

$$\kappa^2(k_z,\omega) = k_z^2 - \frac{\omega^2}{c^2} \ , \tag{22}$$

and $I_\ell(x)$ and $K_\ell(x)$ are modified Bessel functions of the first and second kinds, respectively, of order $\ell$. For our purposes here, the analysis in Sec. III makes the further assumption that

$$|\kappa^2(k_z,\omega)| r_w^2 = \left| k_z^2 - \frac{\omega^2}{c^2} \right| r_w^2 \ll 1 \ . \tag{23}$$

Whenever Eq. (23) is satisfied, Eqs. (15) and (16) can be approximated by $\nabla_\perp^2 \delta\phi = 0 = \nabla_\perp^2 \delta A_z$ in the vacuum region ($r_b < r \leq r_w$) where $\delta n_b = 0$, and the solutions to Eqs. (15) and (16) for $\delta\phi^\ell(r)$ and $\delta A_z^\ell(r)$ are linear combinations of $r^\ell$ and $r^{-\ell}$, where $\ell \geq 1$ is an integer. For example, if we estimate the oscillation frequency by $\omega \simeq k_z V_b$, then Eq. (23) reduces to

$$\frac{|k_z|^2 r_w^2}{\gamma_b^2} \ll 1 \ , \tag{24}$$

where $\gamma_b^{-2} = 1 - V_b^2/c^2$. Therefore, for a long, highly-relativistic charge bunch ($\ell_b \gg r_b$, $\gamma_b \gg 1$), the inequality in Eq. (24) is relatively straightforward to satisfy, even when $r_w \gg r_b$, provided the relativistic mass factor $\gamma_b$ is sufficiently large.

## III. KINETIC STABILITY ANALYSIS

### A. Linearized Vlasov-Maxwell Equations

We now make use of Eqs. (14)–(17) and the assumptions summarized in Sec. 2 to derive a dispersion relation that describes detailed stability properties of the charge bunch. In the present analysis, the equilibrium distribution function in Eq. (2) can be expressed as $f_b^0(r, \mathbf{p}) = F_b(H_\perp)\delta(p_z - \gamma_b m_b \beta_b c)$, where $F_b(H_\perp) = (\hat{n}_b/2\pi\gamma_b m_b)\delta(H_\perp - \hat{T}_{\perp b})$. Because $f_b^0$ has zero axial momentum spread about $p_z = \gamma_b m_b \beta_b c$, we express the perturbed distribution function in the linearized Vlasov equation (14) as $\delta f_b(\mathbf{x}, \mathbf{p}, t) = \delta F_b(\mathbf{x}, \mathbf{p}_\perp, t)\delta(p_z - \gamma_b m_b \beta_b c)$. Integrating Eq. (14) over $p_z$ then gives for the evolution of $\delta F_b(\mathbf{x}, \mathbf{p}_\perp, t)$,

$$\left(\frac{\partial}{\partial t} + V_b \frac{\partial}{\partial z} + \mathbf{v}_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \gamma_b m_b \nu_b^2 \mathbf{x}_\perp \cdot \frac{\partial}{\partial \mathbf{p}_\perp}\right)\delta F_b = \frac{e_b}{\gamma_b m_b}\frac{\partial F_b}{\partial H_\perp}\mathbf{p}_\perp \cdot \nabla_\perp(\delta\phi - \beta_b \delta A_z) \ , \tag{25}$$

where $V_b = \beta_b c = const.$ is the axial velocity of the beam particles. Moreover, consistent with Eqs. (12) and (23), we neglect the terms proportional to $\partial^2/\partial z^2 - c^{-2}\partial^2/\partial t^2$ in Eqs. (15) and (16), and the linearized Maxwell equations for $\delta\phi(\mathbf{x}, t)$ and $\delta A_z(\mathbf{x}, t)$ are approximated by

$$\nabla_\perp^2 \delta\phi = -4\pi e_b \int d^2p \delta F_b \ , \tag{26}$$

and

$$\nabla_\perp^2 \delta A_z = -4\pi e_b \beta_b \int d^2p \delta F_b \ . \tag{27}$$

Here, $\delta n_b(\mathbf{x}, t) = \int d^2p \delta F_b(\mathbf{x}, \mathbf{p}_\perp, t)$ is the perturbed number density of beam particles, and $\int d^2p \ldots = \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \ldots$ .

In the subsequent analysis of Eqs. (25)–(27), it is convenient to introduce the new independent variables $\tau$ and $Z$ (replacing $t$ and $z$) defined by

$$\tau = t - z/V_b \ ,$$

$$Z = z \ . \tag{28}$$

In this case, the perturbation in Eq. (11) can be expressed as

$$\delta\psi(\mathbf{x}, Z, \tau) = \delta\psi^{\ell}(r) \exp[i\ell\theta - i\omega\tau - i(\Omega/V_b)Z] \ , \tag{29}$$

where $\ell = 1, 2, \cdots$, is the azimuthal mode number, $\omega$ is the oscillation frequency, and

$$\frac{\Omega}{V_b} = \frac{(\omega - k_z V_b)}{V_b} \tag{30}$$

is the effective axial wavenumber of the perturbation in the new variables $(Z, \tau)$. The significance of the new 'time' variable $\tau$ in Eq. (28) is evident. We consider the case where the head of the charge bunch passes through $z = 0$ at $t = 0$ with velocity $V_b > 0$. Then $V_b\tau = V_b t - z$ is the distance backwards from the head of the beam (at $V_b t$) to axial position $z = Z$. If the charge bunch experiences a perturbation for $\tau > 0$ with real oscillation frequency $\omega$, it is evident from Eqs. (29) and (30) that $\Omega/V_b$ represents the spatial oscillation and growth (or damping) of the perturbation as a function of axial position $Z$. Furthermore, in terms of the new variables $Z$ and $\tau$, the derivatives $\partial/\partial t$ and $\partial/\partial z$ transform according to

$$\begin{aligned} \frac{\partial}{\partial\tau} &= \frac{\partial}{\partial t} \ , \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial Z} - \frac{1}{V_b}\frac{\partial}{\partial\tau} \ . \end{aligned} \tag{31}$$

Making use of Eq. (31), the linearized Vlasov equation (25) for $\delta F_b(\mathbf{x}_\perp, \mathbf{p}_\perp, Z, \tau)$ simplifies to become

$$\left( V_b \frac{\partial}{\partial Z} + \mathbf{v}_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \gamma_b m_b \nu_b^2 \mathbf{x}_\perp \cdot \frac{\partial}{\partial \mathbf{p}_\perp} \right) \delta F_b = \frac{e_b}{\gamma_b m_b} \frac{\partial F_b}{\partial H_\perp} \mathbf{p}_\perp \cdot \nabla_\perp (\delta\phi - \beta_b \delta A_z) \ , \tag{32}$$

where $\delta\phi(\mathbf{x}_\perp, Z, \tau)$ and $\delta A_z(\mathbf{x}_\perp, Z, \tau)$ are determined self-consistently in terms of $\delta F_b$ from Eqs. (26) and (27). Note in Eq. (32) that the perturbed beam dynamics is determined in terms of the wake function $\delta\psi \equiv \delta\phi - \beta_b \delta A_z$.

The left-hand side of Eq. (32) will be recognized as the *total derivative*, $(V_b d/dZ') \times \delta F_b(\mathbf{x}'_\perp, \mathbf{p}'_\perp, Z', \tau')$, following the particle trajectories $\mathbf{x}'_\perp$ and $\mathbf{p}'_\perp$ in the equilibrium field configuration. Here, the characteristics of the differential operator on the left-hand side of Eq. (32) are the particle orbit equations

$$\begin{aligned} V_b \frac{d}{dZ'} \mathbf{x}'_\perp(Z') &= \mathbf{v}'_\perp(Z') = \frac{1}{\gamma_b m_b} \mathbf{p}'_\perp(Z') \ , \\ V_b \frac{d}{dZ'} \mathbf{p}'_\perp(Z') &= -\gamma_b m_b \nu_b^2 \mathbf{x}'_\perp(Z') \ , \end{aligned} \tag{33}$$

which can be combined to give

$$V_b^2 \frac{d^2}{dZ'^2}\mathbf{x}'_\perp + \nu_b^2 \mathbf{x}'_\perp = 0 \ . \tag{34}$$

In order to solve Eq. (32), the solutions of physical interest to the transverse orbit equations (33) and (34) are those that pass through the phase space point $(\mathbf{x}_\perp, \mathbf{p}_\perp)$ at $Z' = Z$, i.e.,

$$\begin{aligned} \mathbf{x}'_\perp(Z' = Z) &= \mathbf{x}_\perp \ , \\ \mathbf{p}'_\perp(Z' = Z) &= \mathbf{p}_\perp \ . \end{aligned} \tag{35}$$

Solving Eqs. (33) and (34) subject to Eq. (35), we readily obtain

$$\begin{aligned} \mathbf{x}'_\perp(Z') &= \mathbf{x}_\perp \cos[(\nu_b/V_b)(Z'-Z)] + \frac{\mathbf{p}_\perp}{\gamma_b m_b \nu_b} \sin[(\nu_b/V_b)(Z'-Z)] \ , \\ \mathbf{p}'_\perp(Z') &= \mathbf{p}_\perp \cos[(\nu_b/V_b)(Z'-Z)] - \gamma_b m_b \nu_b \mathbf{x}_\perp \cos[(\nu_b/V_b)(Z'-Z)] \ , \end{aligned} \tag{36}$$

where $\mathbf{p}'_\perp = \gamma_b m_b \mathbf{v}'_\perp = \gamma_b m_b V_b d\mathbf{x}'_\perp/dZ'$. As expected, for the flattop density profile in Eq. (4), the transverse orbits in Eq. (36) are oscillatory functions of $Z' - Z$ with wavelength $\lambda_b = 2\pi V_b/\nu_b$, where $\nu_b = (\omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2/2\gamma_b^2)^{1/2}$ is the (depressed) betatron frequency defined in Eq. (6).

The linearized Vlasov equation (32) is now formally integrated using the method of characteristics[1, 37–40]. Expressing the left-hand side of Eq. (32) as $V_b(d/dZ')$ $\times\delta F_b(\mathbf{x}'_\perp, \mathbf{p}'_\perp, Z', \tau')$, we assume spatially amplifying perturbations ($Im\Omega > 0$) and integrate Eq. (32) from $Z' = -\infty$ (where $\delta F_b$ is assumed to be negligibly small) to $Z' = Z$. This gives

$$\begin{aligned} \delta F_b(\mathbf{x}_\perp, \mathbf{p}_\perp, Z, \tau) &= e_b \frac{\partial}{\partial H_\perp} F_b(H_\perp) \\ &\times \int_{-\infty}^{Z} \frac{dZ'}{V_b} \mathbf{v}'_\perp \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} [\delta\phi(\mathbf{x}'_\perp, Z', \tau) - \beta_b \delta A_z(\mathbf{x}'_\perp, Z', \tau)] \ . \end{aligned} \tag{37}$$

Here, use has been made of the fact that $H'_\perp = H_\perp = const.$ is a single-particle constant of the motion ($dH'_\perp/dZ' = 0$) in the equilibrium field configuration. In the integration over $Z'$ on the right-hand side of Eq. (37), $\mathbf{x}'_\perp(Z')$ and $\mathbf{p}'_\perp(Z') = \gamma_b m_b \mathbf{v}'_\perp(Z')$ are the single-particle orbits in Eq. (6) that pass through the phase space point $(\mathbf{x}_\perp, \mathbf{p}_\perp)$ at $Z' = Z$.

For the choice of equilibrium distribution $F_b(H_\perp)$ in Eq. (2), we calculate the perturbed number density $\delta n_b(\mathbf{x}_\perp, Z, \tau) = \int d^2p \delta F_b(\mathbf{x}_\perp, \mathbf{p}_\perp, Z, \tau)$ from Eq. (37) and substitute into Maxwell's equations (26) and (27), which gives closed equations for the perturbed potentials,

$\delta\phi(\mathbf{x}_\perp, Z, \tau)$ and $\delta A_z(\mathbf{x}_\perp, Z, \tau)$. Assuming perturbations of the form in Eq. (24) for $Im\Omega > 0$ and azimuthal mode number $\ell = 1, 2, \cdots$, and carrying out the integration over $Z'$ in Eq. (37), it is found that a class of solutions exists with density perturbation amplitude $\delta n_b^\ell(r) = \int d^2p\delta F_b^\ell(r, \mathbf{p}_\perp)$ localized at the surface of the charge bunch ($r = r_b$). Without presenting algebraic details[1, 38], we obtain

$$4\pi e_b \int d^3p\delta F_b^\ell(r, \mathbf{p}_\perp) = -\frac{2\ell}{r_b}\chi_b^\ell(\Omega)[\delta\phi^\ell(r) - \beta_b\delta A_z^\ell(r)]\delta(r - r_b) \ . \tag{38}$$

Here, the response function $\chi_b^\ell(\Omega)$ is defined by

$$\chi_b^\ell(\Omega) = -\frac{\hat{\omega}_{pb}^2}{2\ell 2^\ell \nu_b^2}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}\frac{(\ell-2m)\nu_b}{\Omega - (\ell-2m)\nu_b} \ , \tag{39}$$

where $\Omega = \omega - k_z V_b$ is the Doppler shifted frequency, $\hat{\omega}_{pb} = (4\pi\hat{n}_b e_b^2/\gamma_b m_b)^{1/2}$ is the relativistic plasma frequency, and $\nu_b = (\omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2/2\gamma_b^2)^{1/2}$ is the depressed betatron frequency. As expected, the response function in Eq. (39) has a rich harmonic content at harmonics of $\nu_b$. We define

$$\delta\psi^\ell(r) = \delta\phi^\ell(r) - \beta_b\delta A_z^\ell(r) \ , \tag{40}$$

and denote $\delta\hat{\psi}^\ell = \delta\psi^\ell(r_b)$, $\delta\hat{\phi}^\ell = \delta\phi^\ell(r_b)$ and $\delta\hat{A}_z^\ell = \delta A_z^\ell(r_b)$. Substituting Eqs. (29), (38) and (40) into Eqs. (26) and (27), Maxwell's equations become

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{\ell^2}{r^2}\right)\delta\phi^\ell(r) = \frac{2\ell}{r_b}\chi_b^\ell(\Omega)\delta\hat{\psi}^\ell\delta(r - r_b) \ , \tag{41}$$

and

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{\ell^2}{r^2}\right)\delta A_z^\ell(r) = \frac{2\ell}{r_b}\beta_b\chi_b^\ell(\Omega)\delta\hat{\psi}^\ell\delta(r - r_b) \ , \tag{42}$$

for azimuthal mode numbers $\ell = 1, 2, \cdots$.

Equations (41) and (42), derived for perturbations about the equilibrium distribution $f_b^0(r, \mathbf{p})$ in Eq. (2) with flattop-density profile in Eq. (4), constitute the final forms of the eigenvalue equations used in the present stability analysis. Here, Eqs. (41) and (42) are to be solved over the interval $0 \le r \le r_w$ for the eigenfunctions $\delta\phi^\ell(r)$ and $\delta A_z^\ell(r)$ and eigenvalue $\Omega$, subject to the condition that $\delta\phi^\ell(r)$ and $\delta A_z^\ell(r)$ be regular at the origin ($r = 0$), and satisfy the boundary conditions in Eq. (20) at the conducting wall ($r = r_w$). It should be emphasized that Eqs. (41) and (42) are valid over the entire range of normalized beam intensity, $0 < s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2 < 1$, subject to the assumption of low-frequency, long-wavelength perturbations in Eqs. (12) and (23).

### B. Derivation of Dispersion Relation

We now solve Eqs. (41) and (42) in the beam interior ($0 \leq r < r_b$) and in the vacuum region outside the charge bunch ($r_b < r \leq r_w$). The solutions to Eqs. (41) and (42) that are regular at $r = 0$ can be expressed as

$$\delta\phi^\ell(r) = \begin{cases} \delta\hat{\phi}^\ell (r/r_b)^\ell \ , & 0 \leq r < r_b \ , \\ A'(r/r_b)^\ell + B'(r_b/r)^\ell \ , & r_b < r \leq r_w \ , \end{cases} \tag{43}$$

and

$$\delta A_z^\ell(r) = \begin{cases} \delta\hat{A}_z^\ell (r/r_b)^\ell \ , & 0 \leq r < r_b \ , \\ A(r/r_b)^\ell + B'(r_b/r)^\ell \ , & r_b < r \leq r_w \ , \end{cases} \tag{44}$$

where $\delta\hat{\phi}^\ell \equiv \delta\phi^\ell(r = r_b)$ and $\delta\hat{A}_z^\ell \equiv \delta A_z^\ell(r = r_b)$, and $A'$, $B'$, $A$ and $B$ are constants. We enforce continuity of $\delta\phi^\ell(r)$ and $\delta A_z^\ell(r)$ at $r = r_b$, which gives

$$\begin{aligned} A' + B' &= \delta\hat{\phi}^\ell \ , \\ A + B &= \delta\hat{A}_z^\ell \ . \end{aligned} \tag{45}$$

Note from Eqs. (41) and (42) that there are surface charge and current perturbations at $r = r_b$ [the terms proportional to $\delta(r - r_b)$]. The remaining boundary conditions at $r = r_b$ are therefore obtained by operating on Eqs. (43) and (44) with $\int_{r_b(1-\epsilon)}^{r_b(1+\epsilon)} dr r \cdots$, and taking the limit $\epsilon \to 0_+$. This readily gives

$$\begin{aligned} A' - B' - \delta\hat{\phi}^\ell &= 2\chi_b^\ell(\Omega)\delta\hat{\psi}^\ell \ , \\ A - B - \delta\hat{A}_z^\ell &= 2\beta_b\chi_b^\ell(\Omega)\delta\hat{\psi}^\ell \ , \end{aligned} \tag{46}$$

where $\delta\hat{\psi}^\ell \equiv \delta\hat{\phi}^\ell - \beta_b\delta\hat{A}_z^\ell$, and $\ell = 1, 2, 3, \cdots$. Equation (46) effectively determines the discontinuity in perturbed radial electric field (azimuthal magnetic field) in terms of the perturbed surface charge density (current density), which is proportional to $\chi_b^\ell(\Omega)$. Solving for the coefficients $A'$, $B'$, $A$ and $B$ in terms of $\delta\hat{\phi}^\ell$ and $\delta\hat{A}_z^\ell$, we obtain from Eqs. (45) and

(46)

$$A' = \delta\hat{\phi}^\ell + \chi_b^\ell(\delta\hat{\phi}^\ell - \beta_b\delta\hat{A}_z^\ell) \ ,$$

$$B' = -\chi_b^\ell(\delta\hat{\phi}^\ell - \beta_b\delta\hat{A}_z^\ell) \ ,$$

$$B = -\beta_b\chi_b^\ell(\delta\hat{\phi}^\ell - \beta_b\delta A_z^\ell) = \beta_b B' \ ,$$

$$A = \delta\hat{A}_z^\ell + \beta_b\chi_b^\ell(\delta\hat{\phi}^\ell - \beta_b\delta\hat{A}_z^\ell) \ , \tag{47}$$

where $\chi_b^\ell(\Omega)$ is defined in Eq. (39), and use has been made of $\delta\hat{\psi}^\ell = \delta\hat{\phi}^\ell - \beta_b\delta\hat{A}_z^\ell$.

We now enforce the boundary conditions at the conducting wall ($r = r_w$) given in Eq. (20). Making use of the solutions for $\delta\phi^\ell(r)$ and $\delta A_z^\ell(r)$ in the vacuum region ($r_b < r \leq r_w$) given in Eqs. (43) and (44), the boundary conditions in Eq. (20) can be expressed as

$$k_z\left[A' + B'\left(\frac{r_b}{r_w}\right)^{2\ell}\right] - \frac{\omega}{c}\left[A + B\left(\frac{r_b}{r_w}\right)^{2\ell}\right] = \frac{i\ell}{r_w}\tilde{Z}(\omega)\left[A - B\left(\frac{r_b}{r_w}\right)^{2\ell}\right] \ , \tag{48}$$

and

$$\frac{\ell}{r_w}\left[A' + B'\left(\frac{r_b}{r_w}\right)^{2\ell}\right] = -ik_z\tilde{Z}(\omega)\left[A - B\left(\frac{r_b}{r_w}\right)^{2\ell}\right] - \frac{i\omega}{c}\tilde{Z}(\omega)\left[A' - B'\left(\frac{r_b}{r_w}\right)^{2\ell}\right] \ , \tag{49}$$

where $\tilde{Z}(\omega)$ is the wall impedance. Equations (47)–(49) can be combined to give two linear, homogeneous equations relating the perturbation amplitudes $\delta\hat{\phi}^\ell$ and $\delta\hat{A}_z^\ell$. The dispersion relation for the complex frequency $\Omega$ is then obtained by setting the determinant of the 2x2 coefficient matrix equal to zero. In the limit of a perfect conductor with $\tilde{Z} \to 0$, note that Eqs. (48) and (49) give $A' \to -B'(r_b/r_w)^{2\ell}$ and $A \to -B(r_b/r_w)^{2\ell}$, which correspond to the boundary conditions for a perfect conductor, $\delta\phi^\ell(r = r_w) = 0 = \delta A_z^\ell(r = r_w)$, as expected. For $\tilde{Z} \neq 0$, it is convenient to express

$$A = -B\left(\frac{r_b}{r_w}\right)^{2\ell}(1 + \Delta) \ ,$$

$$A' = -B'\left(\frac{r_b}{r_w}\right)^{2\ell}(1 + \Delta') \ , \tag{50}$$

and make use of Eqs. (48) and (49) to solve for $\Delta$ and $\Delta'$ in terms of the impedance $\tilde{Z}(\omega)$. Substituting Eq. (50) into Eqs. (48) and (49), and making use of $B = \beta_b B'$ [Eq. (47)], we obtain

$$k_z\Delta' - \beta_b\left(\frac{\omega}{c} + \frac{\ell}{r_w}i\tilde{Z}\right)\Delta = 2\beta_b\frac{\ell}{r_w}i\tilde{Z} \ ,$$

$$\left(\frac{\ell}{r_w} + \frac{\omega}{c}i\tilde{Z}\right)\Delta' + \beta_b k_z i\tilde{Z}\Delta = -2\left(\beta_b k_z + \frac{\omega}{c}\right)i\tilde{Z} \ . \tag{51}$$

Equation (51) can be used to determine closed expressions for $\Delta'$ and $\Delta$ in terms of the wall impedance $\tilde{Z}(\omega)$. For example, if

$$|\tilde{Z}| \ll \left|\frac{\omega}{c}\frac{r_w}{\ell}\right| , \left|\frac{c}{\omega}\frac{\ell}{r_w}\right| , \tag{52}$$

then the approximate solutions to Eq. (51) are given correct to leading order by

$$\begin{aligned} \Delta' &= -2\frac{\omega r_w}{\ell c}\left(1+\frac{k_z V_b}{\omega}\right) i\tilde{Z}(\omega) \ , \\ \Delta &= -2\frac{\ell c}{\omega r_w}\left[1+\frac{k_z^2 r_w^2}{\ell^2}\left(1+\frac{\omega}{k_z V_b}\right)\right] i\tilde{Z}(\omega) \, , \end{aligned} \tag{53}$$

where $V_b = \beta_b c$. If we estimate $\omega \approx k_z V_b$, then the inequalities in Eq. (52) assure that $|\Delta'|$, $|\Delta| \ll 1$ and that the wall impedance contributions proportional to $\Delta$ and $\Delta'$ in Eq. (50) represent small corrections to the results for a perfectly conducting wall.

In any case, we now make use of Eq. (50) to derive the dispersion relation that determines $\Omega$ (generally complex) in terms of the oscillation frequency $\omega$ and system parameters such as the plasma frequency $\hat{\omega}_{pb}$, depressed betatron frequency $\nu_b$, and wall impedance $\tilde{Z}(\omega)$. Substituting Eq. (47) into Eq. (50), where $\Delta$ and $\Delta'$ solve Eq. (51), and making use of $B = \beta_b B'$, we readily obtain

$$\delta\hat{A}_z^\ell - \beta_b \delta\hat{\phi}^\ell - \beta_b\left(\frac{r_b}{r_w}\right)^{2\ell}(\Delta-\Delta')\chi_b^\ell \delta\hat{\psi}^\ell = 0 \ , \tag{54}$$

and

$$\hat{\delta}A_z^\ell + \beta_b\left[1-\left(\frac{r_b}{r_w}\right)^{2\ell}\right]\chi_b^\ell \delta\hat{\psi}^\ell - \beta_b\left(\frac{r_b}{r_w}\right)^{2\ell}\Delta\chi_b^\ell\delta\hat{\psi}^\ell = 0 \ , \tag{55}$$

where $\delta\hat{\psi}^\ell = \delta\hat{\phi}^\ell - \beta_b\delta\hat{A}_z^\ell$. Rewriting $\delta\hat{A}_z^\ell - \beta_b\delta\hat{\phi}^\ell = (1/\gamma_b^2)\delta A_z^\ell - \beta_b\delta\hat{\psi}^\ell$, and eliminating $\delta\hat{A}_z^\ell$ from Eqs. (54) and (55), we obtain

$$D_b^\ell(\Omega)\delta\hat{\psi}^\ell = 0 \ , \tag{56}$$

where $D_b^\ell(\Omega)$ is the dielectric function defined by

$$D_b^\ell(\Omega) = 1 + \frac{1}{\gamma_b^2}\left[1-\left(\frac{r_b}{r_w}\right)^{2\ell}\right]\chi_b^\ell(\Omega) + \left(\frac{r_b}{r_w}\right)^{2\ell}\chi_b^\ell(\Omega)[\beta_b^2\Delta - \Delta'] \ , \tag{57}$$

and use has been made of $\beta_b^2 = 1 - 1/\gamma_b^2$. The condition for a nontrivial solution ($\delta\hat{\psi}^\ell \neq 0$) to Eq. (56) is

$$D_b^\ell(\Omega) = 0 \ . \tag{58}$$

Equation (58) is the final form of the dispersion relation derived from the linearized Vlasov-Maxwell equations (25)–(27) for perturbations about the choice of equilibrium distribution function in Eq. (2) with corresponding flattop density profile in Eq. (4). The dispersion relation (58) is valid for low-frequency long-wavelength perturbations consistent with Eqs. (12) and (23), and can be applied over a wide range of normalized beam intensity $s_b$ in the range $0 < s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2 < 1$. In the definition of $D_b^\ell(\Omega)$ in Eq. (57), the response function $\chi_b^\ell(\Omega)$ is defined in Eq. (39) for general azumithal mode number $\ell = 1, 2, \cdots$, and the quantities $\Delta$ and $\Delta'$ are determined in terms of the wall impedance $\tilde{Z}(\omega)$ from Eq. (51). In circumstances where Eq. (52) is satisfied, $\Delta$ and $\Delta'$ are given approximately by Eq. (53). Making use of Eq. (53) and $\beta_b^2 = 1 - 1/\gamma_b^2$, it is readily shown that

$$\beta_b^2\Delta - \Delta' = -2i\tilde{Z}(\omega)\frac{c\ell}{\omega r_w}\left[\beta_b^2 + \left(k_z^2 - \frac{\omega^2}{c^2}\right)\frac{r_w^2}{\ell^2} - \frac{k_z^2 r_w^2}{\ell^2\gamma_b^2}\right] . \tag{59}$$

For $k_z^2 r_b^2/\gamma_b^2$, $|k_z^2 - \omega^2/c^2|r_w^2 \ll \beta_b^2$ [see also Eqs. (23) and (24)], note that the last two terms in Eq. (59) can be neglected, and Eq. (59) can be approximated by

$$\beta_b^2\Delta - \Delta' = -2i\beta_b^2\frac{c\ell}{\omega r_w}\tilde{Z}(\omega) . \tag{60}$$

## IV. WALL-IMPEDANCE-DRIVEN INSTABILITY FOR DIPOLE-MODE PERTURBATIONS ($\ell = 1$)

The dispersion relation (58) can be used to investigate detailed stability properties for azimuthal mode numbers $\ell = 1, 2, 3, \cdots$. For present purposes, we consider dipole-mode perturbations with $\ell = 1$. In this case, it follows from Eq. (39) that the response function $\chi_b^{\ell=1}(\Omega)$ is given by

$$\chi_b^{\ell=1}(\Omega) = -\frac{\hat{\omega}_{pb}^2/2}{\Omega^2 - \nu_b^2} , \tag{61}$$

where $\nu_b^2 = \omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2/2\gamma_b^2$. Substituting Eq. (61) into Eq. (57), the dispersion relation (58) reduces to

$$D_b^{\ell=1}(\Omega) = 1 - \frac{1}{2\gamma_b^2}\left(1 - \frac{r_b^2}{r_w^2}\right)\frac{\hat{\omega}_{pb}^2}{\Omega^2 - \nu_b^2} - \frac{1}{2}\frac{r_b^2}{r_w^2}(\beta_b^2\Delta - \Delta')\frac{\hat{\omega}_{pb}^2}{\Omega^2 - \nu_b^2} = 0 , \tag{62}$$

for dipole-mode perturbations with $\ell = 1$. Here, $\Delta$ and $\Delta'$ are determined in terms of the wall impedance $\tilde{Z}(\omega)$ from Eq. (51). In the present analysis we approximate $\beta_b^2\Delta - \Delta'$ by Eq. (60) to the required accuracy, and Eq. (62) reduces to

$$\Omega^2 = \omega_{\beta\perp}^2 - \left(\frac{r_b^2}{r_w}\right)^2\frac{\hat{\omega}_{pb}^2}{2\gamma_b^2} - \left(\frac{r_b}{r_w}\right)^2\beta_b^2\hat{\omega}_{pb}^2\frac{c}{\omega r_w}i\tilde{Z}(\omega) , \tag{63}$$

where use has been made of $\nu_b^2 = \omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2/2\gamma_b^2$.

We consider the case where the dipole perturbation has real oscillation frequency $\omega = \omega_0$, and make use of Eq. (63) to determine the complex solutions for $\Omega = \Omega_r + i\Omega_i$. Referring to Eq. (29), keep in mind that the solutions with $(Im\Omega)/V_b = \Omega_i/V_b > 0$ correspond to spatially amplifying perturbations proportional to $\exp[(\Omega_i/V_b)Z]$. Expressing $\tilde{Z}(\omega_0) = \tilde{Z}_r + i\tilde{Z}_i$, we rewrite Eq. (63) as

$$\Omega^2 = X + iY \tag{64}$$

where

$$\begin{aligned} X &= \omega_{\beta\perp}^2 - \left(\frac{r_b}{r_w}\right)^2 \frac{\hat{\omega}_{pb}^2}{2\gamma_b^2} + \left(\frac{r_b}{r_w}\right)^2 \beta_b^2 \hat{\omega}_{pb}^2 \frac{c}{\omega_0 r_w} \tilde{Z}_i \equiv \Omega_0^2 \ , \\ Y &= -\left(\frac{r_b}{r_w}\right)^2 \beta_b^2 \hat{\omega}_{pb}^2 \frac{c}{\omega_0 r_w} \tilde{Z}_r \ . \end{aligned} \tag{65}$$

Solving Eq. (64) for $\Omega = \Omega_r + i\Omega_i$ readily gives

$$\Omega_r = \pm \frac{1}{\sqrt{2}} [X + (X^2 + Y^2)^{1/2}]^{1/2} \ , \tag{66}$$

and

$$\Omega_i = \mp \frac{1}{\sqrt{2}} [-X + (X^2 + Y^2)^{1/2}]^{1/2} \ , \tag{67}$$

where $X = \Omega_0^2 > 0$ is assumed. The solutions in Eq. (66) correspond to sideband oscillations. Representing $k_z = k_{zr} + ik_{zi}$, then $\Omega_r = Re(\omega_0 - k_z V_z) = \omega_0 - k_{zr} V_b$, and the upper (+) and lower (−) signs in Eq. (66) correspond to excitations with phase velocity $\omega_0/k_{zr} > V_b$ and $\omega_0/k_{zr} < V_b$, respectively. Note from Eqs. (66) and (67) that the upper sideband is damped ($\Omega_i < 0$) whereas the lower sideband is growing ($\Omega_i > 0$) whenever $\tilde{Z}_r \neq 0$ ($Y \neq 0$). For the case of sufficiently low wall impedance that $|Y| \ll X = \Omega_0^2$, note that Eqs. (66) and (67) can be approximated by

$$\begin{aligned} \Omega_r &= \pm \Omega_0 \ , \\ \Omega_i &= \mp \frac{1}{2} \frac{|Y|}{\Omega_0} \ , \end{aligned} \tag{68}$$

where $\Omega_0$ and $Y$ are defined in Eq. (65).

We now consider Eqs. (66)–(68) in the three cases corresponding to: (a) perfectly conducting cylindrical wall with $\tilde{Z}(\omega) = 0$; (b) conducting cylindrical wall with conductivity $\sigma$ and $\tilde{Z}(\omega) \neq 0$; and (c) wall with model impedance function $\tilde{Z}(\omega)$.

*(a) Perfectly Conducting Cylindrical Wall* ($\tilde{Z} = 0$)*:* For a perfecftly conducting wall with $\tilde{Z}(\omega) = 0$, Eqs. (66) and (67) reduce to

$$\Omega_r = \pm\omega_{\beta\perp}\left(1 - \frac{r_b^2}{r_w^2}\frac{\hat{\omega}_{pb}^2}{2\gamma_b^2\omega_{\beta\perp}^2}\right)^{1/2} ,$$

$$\Omega_i = 0 . \tag{69}$$

¿From Eq. (69), note that $\Omega_r \simeq \pm\omega_{\beta\perp}$ whenever $s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2 \ll 1$. On the other hand, for a space-charge-dominated beam with $s_b \to 1$, Eq. (69) reduces to $\Omega_r \simeq \pm\omega_{\beta\perp}(1 - r_b^2/r_w^2)^{1/2}$. In general, $\Omega_r^2$ is reduced relative to $\omega_{\beta\perp}^2$ due to image charge effects [the term proportional to $(r_b^2/r_w^2)(\hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2)$ in Eq. (69)].

*(b) Conducting Cylindrical Wall with Conductivity* $\sigma$ ($\tilde{Z} \neq 0$)*:* For a smooth cylindrical wall with electrical conductivity $\sigma$, the impedance function can be approximated by Eq. (21), or equivalently,

$$\tilde{Z}(\omega_0) = \frac{\omega_0}{2c}\delta(1 - i) , \tag{70}$$

where $\delta \equiv 1/(2\pi\sigma\omega_0)^{1/2}$ is the skin depth. Substituting Eq. (70) into Eqs. (65)–(68) gives

$$\Omega_r = \pm\Omega_0 \simeq \pm\omega_{\beta\perp}\left(1 - \frac{r_b^2}{r_w^2}\frac{\hat{\omega}_{pb}^2}{2\gamma_b^2\omega_{\beta\perp}^2}\right)^{1/2} ,$$

$$\Omega_i = \mp\frac{1}{4}\frac{r_b^2}{r_w^2}\beta_b^2\frac{\hat{\omega}_{pb}^2}{\Omega_0}\frac{\delta}{r_w} , \tag{71}$$

where we have neglected $\tilde{Z}_i$ in the definition of $\Omega_0$ in Eq. (65) for $\delta \ll r_w$. Note from Eq. (71) that the lower sideband with $\Omega_r = -\Omega_0$ is unstable ($Im\Omega = \Omega_i > 0$), and that the growth rate is proportional to $\hat{\omega}_{pb}^2/\Omega_0$, which is an increasing function of the beam density $\hat{n}_b$. Moreover, the growth rate $\Omega_i$ is linearly proportional to the normalized skin depth $\delta/r_w$, where $\delta = 1/(2\pi\sigma\omega_0)^{1/2} \to 0$ as $\sigma \to \infty$.

*(c) Wall with Model Impedance:* The interaction of an intense beam with the induction modules of course depends on the cavity design, details of the drive circuitry, etc. This interaction is often modeled by a complex coupling impedance [2, 42, 43] $\tilde{Z}(\omega_0) = \tilde{Z}_r(\omega_0) + i\tilde{Z}_i(\omega_0)$, where

$$\tilde{Z}_r(\omega_0) = \frac{R}{1 + \omega_0^2R^2C^2} ,$$

$$\tilde{Z}_i(\omega_0) = \frac{\omega_0R^2C}{1 + \omega_0^2R^2C^2} . \tag{72}$$

Here, $R$ is the resistance associated with the external drive source, $C$ is the effective module capacitance, and $\omega_0$ is the excitation frequency. Equation (72) can be substituted into Eqs. (65)–(67) to calculate the real frequency $\Omega_r$ and growth rate $\Omega_i$, including the effects of the modification of $\Omega_0^2$ by finite $\tilde{Z}_i$ due to the module capacitance $C$ [see Eqs. (65) and (72)]. For present purposes, we neglect the $\tilde{Z}_i$ contribution to $\Omega_0^2$, and make use of Eq. (68), which gives

$$\Omega_r = \pm\Omega_0 \simeq \pm\omega_{\beta\perp}\left(1 - \frac{r_b^2}{r_w^2}\frac{\hat{\omega}_{pb}^2}{2\gamma_b^2\omega_{\beta\perp}^2}\right)^{1/2} ,$$

$$\Omega_i = \mp\frac{1}{2}\frac{r_b^2}{r_w^2}\beta_b^2\frac{\hat{\omega}_{pb}^2}{\Omega_0}\frac{c}{\omega_0 r_w}\frac{R}{1+\omega_0^2 R^2 C^2} . \tag{73}$$

Note that the growth rate $\Omega_i$ in Eq. (73) is a maximum for $\omega_0^2 R^2 C^2 = 1/2$.

In concluding this section it is evident that the main stability results [Eqs. (67), (68), (71) and (73)] can be applied over a wide range of system parameters and models for the wall impedance $\tilde{Z}(\omega)$. Of particular interest is the scaling of the growth rate $\Omega_i = Im\Omega$ with normalized beam intensity $s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2$. Making use of Eqs. (65) and (68), where the $\tilde{Z}_i$ contribution to $\Omega_0^2$ is neglected in Eq. (65), it is straightforward to show that the growth rate of the lower sideband in Eq. (68) can be expressed as

$$\frac{\Omega_i}{\omega_{\beta\perp}} = \frac{\beta_b^2\gamma_b^2 s_b r_b^2/r_w^2}{(1 - s_b r_b^2/r_w^2)^{1/2}}\left|\frac{e}{\omega_0 r_w}\tilde{Z}_r(\omega_0)\right| . \tag{74}$$

The expression for growth rate in Eq. (74) is valid over the entire range of normalized beam intensity $s_b$ ranging from low-intensity beams with $s_b \ll 1$ to space-charge-dominated beams with $s_b \to 1$. For fixed values of $\beta_b\gamma_b$, $r_b/r_w$ and $\tilde{Z}_r$, note from Eq. (74) that the growth rate $\Omega_i$ is an increasing function of normalized beam intensity $s_b$. Furthermore, for fixed values of $\beta_b\gamma_b$, $s_b$ and $\tilde{Z}_r$, the growth rate $\Omega_i$ increases as the conducting wall is brought into closer proximity to the beam (increasing values of $r_b^2/r_w^2$).

## V. CONCLUSIONS

As noted in Sec. I, there has been considerable recent analytical progress in applying the Vlasov-Maxwell equations to investigate the detailed equilibrium and stability properties of intense charged particle beams. These investigations have included a wide variety of diverse applications ranging from the Harris-like instability driven by large temperature anisotropy with $T_{\perp b} \gg T_{\parallel b}$[37], to the dipole-mode two-stream instability for an intense ion beam propagating through background electrons[38], to the resistive hose instability[39] and the sausage and hollowing instabilities[40] for intense beam propagation through background plasma, to the development of a nonlinear stability theorem[23, 24] in the smooth-focusing approximation. Building on these advances, in the present analysis we have reexamined the classical wall-impedance-driven instability[41–45], making use of the linearized Vlasov-Maxwell equations for perturbations about a Kapchinskij-Vladimirskij (KV) beam equilibrium with flattop density profile, assuming a long charge bunch (bunch length $\ell_b \gg$ bunch radius $r_b$) propagating through a cylindrical pipe with radius $r_w$ and wall impedance $\tilde{Z}(\omega)$. The stability analysis (Secs. II and III) was carried out for perturbations with azimuthal mode number $\ell \geq 1$ about a cylindrical KV beam in the smooth-focusing approximation, leading to the dispersion relation (58). Detailed stability properties were determined (Sec. IV) for dipole-mode perturbations ($\ell = 1$), assuming negligibly small axial momentum spread of the beam particles. A key feature of the present analysis is that the instability growth rate for the dipole mode [Eq. (67)] is valid for general value of the normalized beam intensity $s_b = \hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2$ in the interval $0 < s_b < 1$, where $\hat{\omega}_{pb} = (4\pi\hat{n}_b e_b^2/\gamma_b m_b)^{1/2}$ is the relativistic plasma frequency and $\omega_{\beta\perp}$ is the applied focusing frequency.

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