Influence of profile shape on the diocotron instability in a non-neutral plasma column

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In this paper we examine theoretically the influence of density profile shape on the diocotron instability in a cylindrical, low-density \((\omega_{pe}^2 \ll \omega_{ce}^2)\) non-neutral electron plasma column confined radially by a uniform axial magnetic field \(B_0 \hat{e}_z\). The analysis assumes electrostatic flute perturbations \((\partial / \partial z = 0)\) about an axisymmetric equilibrium density profile \(n_e^0(r)\), where \(r = (x^2 + y^2)^{1/2}\) is the radial distance from the column axis. Two classes of density profiles with inverted population in radius \(r\) are considered. These are the following: (a) a *step-function* density profile with uniform density \(\bar{n}_e\) \(\Delta\) in the column interior \(0 \leq r < r_b^-\), and uniform density \(\bar{n}_e\) in an outer annular region \(r_b^- < r < r_b^+\); and (b) a *continuously-varying* density profile of the form \(n_e(r) = \bar{n}_e(\Delta + r^2/r_b^+)^{1/2}\) over the interval \(0 \leq r < r_b^-\). Here, \(\bar{n}_e\), \(r_b^-\), \(r_b^+\) are positive constants, and the dimensionless parameter \(\Delta\) measures the degree of “hollowness” of the equilibrium density profile \(n_e^0(r)\). Detailed linear stability properties are calculated for a wide range of system parameters, including values of the “filling factor” \(\Delta\), radial location \(r_w\) of the cylindrical conducting wall, azimuthal mode number \(l\), etc. As a general remark, in both cases, it is found that small increases in \(\Delta\) from the value \(\Delta = 0\) (corresponding to the strongest diocotron instability) can have a large effect on the growth rate and detailed properties of the instability. In addition, for the step-function density profile, the instability tends to be algebraic in nature and have a large growth rate in the unstable region of parameter space, whereas for the continuously-varying density profile, the instability is typically much weaker and involves a narrow class of resonant particles at radius \(r = r_s\) satisfying the resonance condition \(\omega_e - \omega_{pe}(r_s) = 0\). Here, \(\omega_e\), \(Re\ \omega\) is the real oscillation frequency, and \(\omega_{pe}(r) = -E_0(r)/rB_0\) is the equilibrium \(E^0 \times B_0 \hat{e}_e\) rotation velocity of the plasma column.

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I. INTRODUCTION

The diocotron instability,\(^1\) first examined theoretically by MacFarlane and Hay,\(^2\) and Levy *et al.*,\(^3\)–\(^5\) and observed in early experiments by Kyhl and Webster,\(^6\)–\(^7\) and Kapetanakos *et al.*,\(^8\) is perhaps the most ubiquitous instability in a low-density \((\omega_{pe}^2 \ll \omega_{ce}^2)\) non-neutral electron plasma column confined radially by a uniform axial magnetic field \(B_0 \hat{e}_z\). To briefly summarize, the diocotron instability is driven by a sufficiently strong shear in the angular \(E^0 \times B_0 \hat{e}_e\) rotation velocity, \(\omega_{pe}(r) = -E_0(r)/rB_0\), of the plasma column. Here, \(r = (x^2 + y^2)^{1/2}\) is the radial distance from the column axis, and \(E_0(r)\) is the equilibrium radial electric field determined self-consistently from Poisson’s equation in terms of the equilibrium density profile \(n_e^0(r)\) (assumed axisymmetric). Whenever the density profile has an inverted population as a function of the radial coordinate \(r\) (an off-axis density maximum), the sign of \(\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} [r^2 \omega_e(r)] \right]\) changes over the radial extent of the plasma column, and the shear in the angular flow velocity can provide the free energy to drive the Kelvin–Helmholtz-like instability known as the *diocotron instability*. Indeed, it can be shown that a *sufficient condition for stability*\(^9\)–\(^12\) for small-amplitude electrostatic flute perturbations \((\partial / \partial z = 0)\) is that \(n_e^0(r)\), or equivalently, \(r^{-1}(\partial \omega_e/\partial r) [r^2 \omega_e(r)]\), be a monotonically decreasing function of radius \(r\). While detailed electrostatic stability properties have been calculated theoretically for a few simple choices of density profile \(n_e^0(r)\), such as a hollow step-function annulus,\(^1\) or weak resonant versions\(^9\) of the diocotron instability, there has not been a systematic analysis of properties of the diocotron instability as a function of the shape of the density profile \(n_e^0(r)\). Nonetheless, over the past decade, experimental studies\(^13\)–\(^18\) of the diocotron instability, and related investigations of diocotron-like modes and vortex formation and merging have become increasingly sophisticated. Therefore, in the present analysis, we present a systematic analysis of the electrostatic eigenvalue equation\(^1\) to determine the detailed influence of profile shape on the diocotron instability, at least for two classes of equilibrium density profiles \(n_e^0(r)\) with an inverted population in radius \(r\).

The organization of this paper is the following. The assumptions and theoretical model are discussed in Sec. II, and detailed stability properties are calculated in Sec. III for two choices of equilibrium density profile \(n_e^0(r)\). These are the following: (a) a *step-function* density profile [Eq. (7)] with uniform density \(\bar{n}_e\) \(\Delta\) in the column interior \(0 \leq r < r_b^-\), and uniform density \(\bar{n}_e\) in an outer annular region \(r_b^- < r < r_b^+\); and (b) a *continuously-varying* density...
profile [Eq. (9)] of the form \( n_0^b(r) = \hat{n}_0(\Delta + r^2/r_b^2)(1 - r^2/r_b^2) \) over the interval \( 0 \leq r < r_w \). Here, \( \hat{n}_0, r_b^- \), \( r_b^+ \) and \( r_b \) are constant polars, a perfectly conducting wall is located at radius \( r = r_w \) and the dimensionless parameter \( \Delta \) measures the degree of ‘hollowness’ of the equilibrium density profile \( n_0^b(r) \). Detailed linear stability properties are calculated in Sec. III for a wide range of system parameters, including values of the ‘filling factor’ \( \Delta \), radial location \( r_w \) of the cylindrical conducting wall, azimuthal mode number \( l \), etc. As a general remark, in both cases, it is found that small increases in \( \Delta \) from the value \( \Delta = 0 \) (corresponding to the strongest diocotron instability) can have a large effect on the growth rate and detailed properties of the instability. In addition, for the step-function density profile in Eq. (7), the instability tends to be algebraic in nature and have a large growth rate in the unstable region of parameter space, whereas for the continuously-varying density profile in Eq. (9), the instability is typically much weaker and involves a narrow class of resonant particles at radius \( r = r_w \) satisfying the resonance condition \( \omega_r - i \omega_p(r) = 0 \) in Eq. (42). Here, \( \omega_r = \text{Re} \omega \) is the real oscillation frequency. As a final point, although the present analysis is restricted to the diocotron instability for low-density non-uniform plasma with \( \omega_p^2(r) < \omega_e^2 \), it should be emphasized that detailed stability behavior and mode oscillation properties also exhibit a sensitive dependence on density profile shape at conditions approaching Brillouin flow \((\omega_p^2/r_e^2 - 1)\) in magnetically-insulated diode geometry.\(^{19,20}\) In this case, as shown by Kaup and Thomas,\(^{20}\) the frequency characteristics of the magnetron mode are modified significantly when the density profile differs from a simple step-function profile.

II. ASSUMPTIONS AND THEORETICAL MODEL

We consider here a cylindrical low-density \((\omega_e^2/\omega_p^2)\) non-neutral electron plasma confined radially by a uniform axial magnetic field \( B_0 \hat{z} \). Equilibrium properties are assumed to be azimuthally symmetric \((\partial / \partial \theta = 0)\) about the column axis and have negligible spatial variation in the axial direction \((\partial / \partial z = 0)\). For low-frequency electrostatic flow perturbations of the form \( \delta \Phi(r, \theta, t) = \Sigma \delta \Phi_i(r) \exp[\text{i}(\theta \theta - \omega t)] \), the eigenvalue equation can be expressed as

\[
1 - \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{r \partial \omega_p^2} \frac{\partial}{\partial r} \delta \Phi_i - \frac{1}{\omega_p^2} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \delta \Phi_i = 0
\]  

(1)

Here, \( l \) is the azimuthal mode number, \( \omega_e = eB_0/m_e c \) is the electron cyclotron frequency and \( \omega_p^2(r) = 4\pi n_0^b(r) e^2/m_e \) is the electron plasma frequency-squared, where \( n_0^b(r) \) is the equilibrium electron density profile, and \( r = (x^2 + y^2)^{1/2} \) is the radial distance from the axis of symmetry. In Eq. (1) and related definitions, \(-e, m_e \) and \( c \) are the electron charge, electron mass and speed of light in vacuo, respectively, and \( \omega_e(r) = -cE^r_1(r)/rB_0 \) is the equilibrium \( E^r_1 \times B_0 \hat{z} \) angular rotation velocity determined self-consistently in terms of \( n_0^b(r) \) from the equilibrium Poisson equation.
In circumstances where \( \partial \omega^2 / \partial r \) does not change sign over the interval \( 0 \leq r \leq r_w \), the only solution to Eq. (6) corresponds to a zero growth rate with \( \gamma = \text{Im} \omega = 0 \). By the same token, it follows from Eq. (6) that a necessary condition for instability (solutions with \( \gamma = \text{Im} \omega > 0 \)) is that \( \partial \omega^2 / \partial r \) change sign over the interval \( 0 \leq r \leq r_w \).

The present stability analysis of Eq. (1) (Sec. III) focuses on two classes of equilibrium profiles with inverted population. First is the \textit{step-function} density profile specified by (Fig. 1)

\[
 n_e^0(r) = \begin{cases} 
 \Delta \cdot \hat{n}_e = \text{const}, & 0 < r < r_b^- \\
 \hat{n}_e = \text{const}, & r_b^- < r < r_b^+ \\
 0, & r_b^+ < r \leq r_w.
\end{cases}
\] 

Here, the (positive) dimensionless parameter \( \Delta \) is a measure of the electron density depression inside the annulus \( r_b^- < r < r_b^+ \), with \( \Delta = 1 \) corresponding to a flat density profile extending from \( r = 0 \) to \( r = r_b^+ \), and \( \Delta = 0 \) corresponding to zero electron density in the region \( 0 < r < r_b^- \). An important physical quantity is the number of electrons per unit axial length of the plasma column defined by \( N_e = 2 \pi \int_0^r \rho n_e^0(r) \). For the equilibrium density profile in Eq. (7) it is readily shown that

\[
 N_e = \pi r_b^2 \hat{n}_e \left[ 1 - (1 - \Delta) \left( \frac{r_b^+}{r_b^-} \right)^2 \right].
\] 

In Sec. III, it will be useful to eliminate \( \hat{n}_e \) in favor of \( N_e \), and examine stability properties for a fixed amount of plasma \( N_e \) but variable profile shape parameters \( \Delta, r_b^-, r_b^+ \), and \( r_b^-/r_b^+ \).

The second class of equilibrium profiles considered in Sec. III has continuous density variation over the radial extent of the plasma column. In particular, we consider the \textit{continuously-varying} density profile \( n_e^0(r) \) specified by (Fig. 2)

\[
 n_e^0(r) = \begin{cases} 
 \hat{n}_e \left( \Delta + \frac{r^2}{r_b^+} \right) \left[ 1 - \frac{r^2}{r_b^+} \right]^2, & 0 < r < r_b^- \\
 0, & r_b^- < r \leq r_w.
\end{cases}
\] 

Here, \( \hat{n}_e \Delta \) is the density on-axis \( (r = 0) \). For \( \Delta \geq 1/2 \), it is readily shown from Eq. (9) that \( n_e^0(r) \) decreases monotonically over the entire interval from \( r = 0 \) to \( r = r_b^- \). On the other hand, for \( 0 \leq \Delta < 1/2 \) (the case of interest here), the density profile in Eq. (9) increases monotonically from the value \( n_e^0 \Delta \) at \( r = 0 \), to the maximum value \( n_{\max} = (4/27) \hat{n}_e (1 + \Delta)^3 \) at radius \( r = r_b^- = (1 - 2\Delta/3)^{1/2} r_b^- \), and then decreases monotonically to zero \( (r = r_b^-) \) over the interval \( r_b^- < r \leq r_b^+ \). The density profile in Eq. (9) is illustrated in Fig. 2 for the case where \( \Delta = 0 \). Finally, making use of \( N_e = 2 \pi \int_0^r \rho n_e^0(r) \), it is readily shown from Eq. (9) that

\[
 N_e = \pi r_b^2 \hat{n}_e \left[ \frac{\Delta}{3} + \frac{1}{12} \right],
\] 

which can be used to express \( \hat{n}_e \) in terms of \( N_e, \Delta \), and \( r_b \).

Finally, for future reference, it will be useful in the analysis of the eigenvalue equation (1) to measure the (complex) eigenfrequency \( \omega \) in units of the (real) frequency \( \omega_1 \) for the \( l = 1 \) dipoecoton mode which is independent of the detailed profile shape for \( n_e^0(r) \). Indeed, as first demonstrated by Levy, for azimuthal mode number \( l = 1 \) and general density profile \( n_e^0(r) \), Eq. (1) supports a stable oscillatory solution \( \text{Im} \omega = 0 \) with an eigenfunction given exactly by \( \delta \Phi_{l=1} = \text{const} \times [\omega - \omega_1(r)] \) over the interval \( 0 \leq r < r_w \). Enforcing the boundary condition \( \delta \Phi_{l=1} \) \( r = r_w = 0 \) then gives \( \omega = \omega_1(r = r_w) = \omega_1 \), where

\[
 \omega_1 = \frac{2 e c}{B_0 r_w} N_e.
\] 

Note from Eq. (11) that \( \omega_1 \) is independent of the detailed profile shape, and depends only on the total amount of plasma \( (N_e) \), the magnetic field strength \( (B_0) \) and the conducting wall radius \( (r_w) \).

III. ANALYSIS OF EIGENVALUE EQUATION

We now make use of the eigenvalue equation (1) to investigate detailed electrostatic stability properties for the equilibrium density profiles \( n_e^0(r) \) in Eqs. (7) and (9), both of which have inverted profiles and are expected to exhibit instability \( (\text{Im} \omega > 0) \), at least for modest values of the ‘‘filling factor’’ \( \Delta \) (see Figs. 1 and 2).
A. Step-function density profile

For the step-function density profile in Fig. 1 and Eq. (7), we first evaluate the angular velocity profile \( \omega(r) \). Substituting Eq. (7) into Eq. (2) readily gives

\[
\omega(r) = \begin{cases} 
\Delta \omega_d, & 0 \leq r < r_b^-, \\
\frac{\omega_d}{1 - (1 - \Delta) \left( \frac{r_b^-}{r} \right)^2}, & r_b^- < r < r_b^+,
\end{cases}
\]

over the radial extent \( 0 \leq r < r_b^+ \) of the plasma column. Here, \( \omega_d \) is an effective diocotron frequency defined in terms of \( n_e \) by

\[
\omega_d = \frac{2}{r} \frac{d n_e}{d r} = \frac{2 \pi n_e c}{B_0}.
\]

Eliminating \( n_e \) in favor of \( N_e \) and \( \omega_1 \) by means of Eqs. (8) and (11), note that \( \omega_d \) can be expressed directly in terms of \( \omega_1 \) according to

\[
\omega_d = \omega_1 \left( \frac{r_w}{r_b^+} \right)^2 \left[ 1 - (1 - \Delta) \left( \frac{r_b^-}{r_b^+} \right)^2 \right].
\]

In the special case where \( \Delta = 1 \) (uniform density plasma column), it is clear from Eq. (12) that \( \omega(r) = \omega_d \) const across the entire radial extent of the plasma column \( 0 \leq r < r_b^+ \). On the other hand, for \( 0 < \Delta < 1 \), there can be a strong radial shear in \( \omega(r) \), particularly when \( \Delta \) is sufficiently small, which leads to unstable solutions to Eq. (1) with \( Im \omega > 0 \).

Substituting the step-function density profile (7) into the eigenvalue equation (1) gives

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \delta \Phi \right) + \frac{\omega_d}{\omega_1} \delta \Phi = -\frac{\omega_d}{\omega_1} \left( \frac{r_b^-}{r} \right)^2 \delta \Phi - \frac{2}{r_b^-} \delta \Phi \delta \left( r - r_b^- \right)
\]

where \( \delta \Phi \) is defined in Eq. (13), and use has been made of Eq. (7) to express \( \delta n_e(r) / \partial r = \delta _n \omega_1 (1 - \Delta) \delta (r - r_b^-) - \delta_n \omega_1 \delta (r - r_b^+) \). For the step-function density profile in Eq. (7), we note from the right-hand side of Poisson’s equation (15) that the perturbed charge density \( -e \delta n_e(r) \) is equal to zero except at the surfaces located at \( r = r_b^- \) and \( r = r_b^+ \), where \( \delta n_e(r) / \partial r \) is singular. Equation (15) is readily solved in the three regions: \( 0 \leq r < r_b^- \) (Region I); \( r_b^- < r < r_b^+ \) (Region II); and \( r_b^+ < r \leq r_w \) (Region III). Denoting \( \delta \Phi_- = \delta \Phi(r = r_b^-) \) and \( \delta \Phi_+ = \delta \Phi(r = r_b^+) \), and enforcing regularity of \( \delta \Phi(r) \) at \( r = 0 \), continuity of \( \delta \Phi(r) \) over the interval \( 0 \leq r \leq r_w \), and \( \delta \Phi(r = r_w) = 0 \) at the conducting wall, we readily obtain the solution to Eq. (15) in the three regions. We find

\[
\delta \Phi_+(r) = \frac{2 \omega_d}{(r_b^-)^2} \left( \frac{r_b^-}{r} \right)^2 \delta \Phi_- - \frac{2 \omega_d}{(r_b^-)^2} \delta \Phi_+ = 0,
\]

where \( \delta \Phi_- = \delta \Phi(r = r_b^-) \) and \( \delta \Phi_+ = \delta \Phi(r = r_b^+) \), and enforcing regularity of \( \delta \Phi(r) \) at \( r = 0 \), continuity of \( \delta \Phi(r) \) over the interval \( 0 \leq r \leq r_w \), and \( \delta \Phi(r = r_w) = 0 \) at the conducting wall, we readily obtain the solution to Eq. (15) in the three regions. We find

\[
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\]

where \( \delta \Phi_- = \delta \Phi(r = r_b^-) \) and \( \delta \Phi_+ = \delta \Phi(r = r_b^+) \), and enforcing regularity of \( \delta \Phi(r) \) at \( r = 0 \), continuity of \( \delta \Phi(r) \) over the interval \( 0 \leq r \leq r_w \), and \( \delta \Phi(r = r_w) = 0 \) at the conducting wall, we readily obtain the solution to Eq. (15) in the three regions. We find

\[
\delta \Phi_+(r) = \frac{2 \omega_d}{(r_b^-)^2} \left( \frac{r_b^-}{r} \right)^2 \delta \Phi_- - \frac{2 \omega_d}{(r_b^-)^2} \delta \Phi_+ = 0.
\]
Equations (21) and (22) are of the form $a\delta \Phi_- + b\delta \Phi_+ = 0$ and $c \delta \Phi_- + d\delta \Phi_+ = 0$, where $a$, $b$, $c$ and $d$ are constant coefficients that depend on $\omega$. The condition for a nontrivial solution with $\delta \Phi_- \neq 0$ and $\delta \Phi_+ \neq 0$ is given by $ad-bc=0$, which plays the role of a dispersion relation that determines the complex oscillation frequency $\omega = \omega_r + i\gamma$ as a function of $\hat{\omega}_d$, $\Delta$, $r_b/r_b^+$, etc.

A simple limit in which to check Eqs. (21) and (22) is the special case where $\Delta = 1$, which corresponds to a uniform density profile with density $\hat{n}_e$ extending from $r=0$ to $r=r_b^+$. Equation (21) gives $\delta \Phi_+ = (r_b^+/r_b^+)^2 \delta \Phi_+$ for $\Delta = 1$, and substitution into Eq. (22) then gives the simple result

$$\omega - (l-1)i\omega_d = \hat{\omega}_d(r_b^+/r_w)^{2l}, \quad \text{for } \Delta = 1. \quad (23)$$

Equation (23) is the expected result for the case of a uniform step-function density profile extending from $r=0$ to $r=r_b^+$. Note from Eq. (23) that $\omega$ is purely real, corresponding to a stable oscillation. Moreover, for $l=1$ and $\Delta = 1$, Eq. (23) reduces to $\omega = \omega_d(r_b^+/r_w)^2 = \omega_1$, as expected [see Eq. (14)].

We now return to Eqs. (21) and (22) for general values of $\Delta$, $r_b^+/r_b^+$, etc. Setting the two-by-two determinant of the coefficients of $\delta \Phi_-$ and $\delta \Phi_+$ in Eqs. (21) and (22) equal to zero, and rearranging terms, it is readily shown that

$$\left(\frac{r_b^+}{r_b^+}\right)^{2l} + \frac{(1-\Delta)(1-(r_b^+/r_b^+)^2)}{\omega - l\hat{\omega}_d} - 1 \times \frac{\hat{\omega}_d[1-(r_b^+/r_b^+)^{2l}]}{\omega - l\hat{\omega}_d[1-(1-\Delta)(r_b^+/r_b^+)^2]} + \frac{1-(r_b^+/r_w)^{2l}}{1-(r_b^+/r_w)^2} = 0. \quad (24)$$

Equation (24) can be cast into the form of a quadratic equation for $\omega/\hat{\omega}_d$. Some straightforward algebraic manipulation gives

$$\left(\frac{\omega}{\hat{\omega}_d}\right)^2 - 2\hat{b}(\omega/\hat{\omega}_d) + \hat{c} = 0, \quad (25)$$

where the constants $\hat{b}$ and $\hat{c}$ are defined by

$$2\hat{b} = l[1+\Delta -(1-\Delta)(r_b^+/r_b^+)^2] - [1-(r_b^+/r_w)^2],$$

$$\hat{c} = l^2[1-(1-\Delta)(r_b^+/r_b^+)^2]\Delta - l[1-(r_b^+/r_w)^2]\Delta$$

$$- [1-(r_b^+/r_w)^2][(1-\Delta)(r_b^+/r_b^+)^2]$$

$$+ (1-\Delta)[1-(r_b^+/r_b^+)^2][1-(r_b^+/r_w)^2]. \quad (26)$$

The solutions to Eq. (25) are given by

$$\frac{\omega}{\hat{\omega}_d} = \hat{b} \pm (\hat{b}^2 - \hat{c})^{1/2}. \quad (28)$$

Evidently, the necessary and sufficient condition for instability is

$$\hat{c} > \hat{b}^2. \quad (29)$$

Whenever the inequality in Eq. (29) is satisfied, the solutions to Eq. (28) occur in conjugate pairs, and the growth rate of the unstable branch is $\text{Im } \omega = + (\hat{c} - \hat{b}^2)^{1/2} \hat{\omega}_d > 0$, and the real frequency is $\text{Re } \omega = \hat{b} \hat{\omega}_d$. Of course, $\text{Im } \omega$ and $\text{Re } \omega$ can be expressed in units of the $l=1$ frequency $\omega_1$ by means of Eq. (14). For azimuthal mode number $l=1$, and general values of $\Delta$, $r_b^+/r_b^+$ and $r_b^+/r_w$, it is readily shown that the two solutions in Eq. (28) reduce to stable oscillations with frequency $\omega = \omega_1(r_b^+/r_b^+)^2$ (upper sign) and $\omega = \omega_1$ (lower sign), where $\omega_1$ has been eliminated in favor of $\omega_1$ by means of Eq. (14).

Typical numerical results for the unstable (upper) branch in Eq. (28) are illustrated in Figs. 3–5. Here, $\text{Re } \omega/\omega_1$ and $\text{Im } \omega/\omega_1$ are plotted versus $r_b^+/r_b^+$ for fixed values of the conducting wall location $(r_b^+/r_w=0.5)$ and values of filling factor $\Delta = 0$ [see Eq. (7)].
factor $\Delta$ corresponding to $\Delta = 0$ (Fig. 3), $\Delta = 0.1$ (Fig. 4) and $\Delta = 0.5$ (Fig. 5). The results in Figs. 3–5 are presented for azimuthal mode numbers $l = 1, 2, \ldots, 10$. For $\Delta = 0$, as expected from previous analyses, it is evident from Fig. 3(b) that as the layer thickness is decreased (increasing values of $r_b^+ / r_b^-$), the $l = 2$ mode is the first to go unstable, then the $l = 3$ mode, then $l = 4$, etc. Furthermore, the maximum growth rate is larger for larger $l$ values, and occurs in the limit of a very thin annulus ($r_b^+ / r_b^- \rightarrow 1$).

Introducing even a small population of electrons in the interior region $0 \leq r < r_b^-$ can have a significant influence on stability properties. This is evident from Fig. 4, where $Re\omega / \omega_1$ and $Im\omega / \omega_1$ are plotted versus $r_b^- / r_b^+$ for the case $\Delta = 0.1$. Comparing Figs. 4(b) and 3(b), several points are noteworthy. First, for $\Delta = 0.1$, the $l = 2$ mode is not unstable. Second, the maximum growth rates are reduced in Fig. 4(b) relative to Fig. 3(b). Finally, the bandwidth structure in Fig. 4(b) differs from that in Fig. 3(b), with each mode stabilizing ($Im\omega = 0$) for $r_b^- / r_b^+$ exceeding a certain critical value less than unity. For example, the $l = 3$ mode in Fig. 4(b) is unstable only in the range $0.47 < r_b^- / r_b^+ < 0.88$, and no longer extends to values of $r_b^- / r_b^+$ approaching unity as in Fig. 3(b). The stabilizing influence of introducing plasma in the interior region $0 \leq r < r_b^-$ is even more strongly evident in Fig. 5, where $Re\omega / \omega_1$ and $Im\omega / \omega_1$ are plotted versus $r_b^- / r_b^+$ for the case where $\Delta = 0.5$.

Detailed stability properties are readily calculated from

FIG. 4. Plots of (a) normalized real frequency $Re\omega / \omega_1$ and (b) normalized growth rate $Im\omega / \omega_1$ versus $r_b^+ / r_b^+$ for the unstable (upper) branch in Eq. (28). Numerical results are presented for azimuthal mode numbers $l = 1, 2, \ldots, 10$, assuming a fixed conducting wall radius with $r_b^+ / r_w = 0.5$, and filling factor $\Delta = 0.1$ [see Eq. (7)].

FIG. 5. Plots of (a) normalized real frequency $Re\omega / \omega_1$ and (b) normalized growth rate $Im\omega / \omega_1$ versus $r_b^+ / r_b^+$ for the unstable (upper) branch in Eq. (28). Numerical results are presented for azimuthal mode numbers $l = 1, 2, \ldots, 10$, assuming fixed conducting wall radius with $r_b^+ / r_w = 0.5$, and filling factor $\Delta = 0.5$ [see Eq. (7)].
Eq. (28) as a function of the filling factor $\Delta$ for fixed values of the geometric factors $r_b^2/r_b$ and $r_b^1/r_w$.

Typical numerical results are presented in Figs. 6 and 7, where $Re\,\omega/\omega_1$ and $Im\,\omega/\omega_1$ are plotted versus $\Delta$ for azimuthal mode numbers $l = 1, 2, \ldots, 10$, and fixed geometric factors $r_b^2/r_b = 0.5$ and $r_b^1/r_w = 0.5$ [see Eq. (7)].

The strong dependence of detailed stability behavior on $\Delta$ is evident from Figs. 6 and 7. First, at fixed values of $r_b^2/r_b$ and $r_b^1/r_w$, but varying $\Delta$, the modes are isolated from one another except for a modest overlap of the $l=2$ and $l=3$ modes over a relatively narrow range of $\Delta$. For example, in Fig. 6(b), when $\Delta = 0.5$, only the $l=4$ mode is unstable, etc. Second, it is clear from Fig. 6(b) that the maximum growth rate is a rapidly decreasing function of $l$ for azimuthal mode numbers $l \geq 3$. Third, comparing Figs. 6(b) and 7(b), it is clear for each value of $l$ that the maximum growth rate is reduced as the conducting wall is brought into closer proximity to the outer surface $r_w^1$ of the plasma. Here, keep in mind that $r_b^1/r_w = 0.5$ in Fig. 6, whereas $r_b^1/r_w = 0.7$ in Fig. 7.

Comparing the relative magnitudes of the $l=2$ and $l=3$ growth rates in Figs. 6(b) and 7(b), it is evident that the $l=2$ mode exhibits a special sensitivity to the location of the conducting wall. Indeed, careful examination of Eqs. (26)–(28) shows that as the conducting wall is removed to infinity ($r_b^1/r_w \rightarrow 0$), the growth rate of the $l=2$ mode reduces exactly to $Im\,\omega = 0$ for arbitrary values of $\Delta$ and $r_b^1/r_w$. On the other hand, at fixed values of $\Delta$ and $r_b^1/r_w$, the higher
mode numbers \( l \geq 3 \) continue to exhibit instability as the conducting wall is removed to infinity with \( r_b^+/r_w \to 0 \), at least in certain very narrow regions of the parameter space \( \Delta, r_b^+/r_w \).

It is evident from Figs. 3–7 that detailed stability behavior exhibits a sensitive dependence on the dimensionless parameters \( \Delta, r_b^+/r_w \) and \( r_b^+/r_w \) for the choice of step-function density profile in Eq. (7). Moreover, the inequality \( \hat{c} > \hat{b}^2 \) in Eq. (29) is a necessary and sufficient condition for instability. Indeed the inequality \( \hat{c} = \hat{b}^2 \) can be used to generate contour plots in two-dimensional subspaces of the parameter space \( (r_b^+/r_w, r_b^+/r_w, \Delta) \) that separate regions of instability \( (\hat{c} > \hat{b}^2 \text{ and } \text{Im}\omega > 0) \) from regions of stable oscillations \( (\hat{b}^2 > \hat{c} \text{ and } \text{Im}\omega > 0) \). Typical numerical results are illustrated in Figs. 8–10, where stability–instability contours are plotted in the parameter space \( (r_b^+/r_w, r_b^+/r_w, \Delta) \). Because \( r_b^+/r_w = r_b^+ \), only the regions above the 45-degree line in Figs. 8–10 are physically allowed. Figure 8 corresponds to the case \( \Delta = 0 \) first considered by Levy, and the contours are plotted for azimuthal mode numbers \( l = 2, 3 \), and 4. Figure 8 is, of course, consistent with the stability behavior presented in Fig. 3. For example, at fixed value of \( r_b^+/r_w \), as 

\[
\frac{r_b^+/r_w}{r_b^+/r_w} = (r_b^+/r_w)(r_w/r_b^+),
\]

is increased, it follows from Fig. 8 that the \( l = 2 \) mode is the first to go unstable, then the \( l = 3 \) mode, then \( l = 4 \), etc., which should be compared with the results in Fig. 3(b). As is evident from Figs. 9 and 10, and as would be expected from the quantitative stability results presented in Figs. 4–7, the stability–instability contours undergo a dramatic change in topology as \( \Delta \) is increased from the \( \Delta = 0 \) case shown in Fig. 8. This is illustrated for azimuthal mode number \( l = 2 \) in Fig. 9 and for \( l = 3 \) in Fig. 10, for several values of the filling factor \( \Delta = 0.1, 0.2, 0.3 \), and 0.6. Comparing the \( l = 2 \) contours in Figs. 8 and 9, it is clear that the \( l = 2 \) contour detaches from the 45-degree line \( (r_b^+/r_w) \) when \( \Delta \neq 0 \), leading to regions of stability \( (\text{Im}\omega = 0) \) at both smaller and larger values of \( r_b^+/r_w \). It is the regions inside the elongated loops in Fig. 9 that correspond to instability with \( \text{Im}\omega > 0 \). Furthermore, from Fig. 9, the area of \( (r_b^+/r_w, r_b^+/r_w) \) parameter space corresponding to instability \( (\text{Im}\omega > 0) \) becomes smaller and smaller as \( \Delta \) is increased, and shifts to larger values of \( r_b^+/r_w \). For azimuthal mode number \( l = 3 \), it is evident from Fig. 10 that the stability–instability contour also detaches from the 45-degree line when \( \Delta \neq 0 \), although the shape and orientation of the unstable region is more complex than for the \( l = 2 \) case shown in Fig. 9.

In concluding Sec. III A, we summarize briefly properties of the (complex) eigenfunction solution for \( \delta\Phi(r) \) in Eq. (16). Here, the amplitude factors \( \delta\Phi_\pm = \delta\Phi_\pm(r = r_b^+) \) and \( \delta\Phi_\pm = \delta\Phi_\pm(r = r_b^+) \) are related by Eqs. (21) and (22), where the complex oscillation frequency \( \omega = \omega_\pm + i\gamma \) is determined self-consistently from the dispersion relation in Eq. (24), or equivalently, Eq. (25). Because of the \( r^{\pm l} \) dependences, we note from Eq. (16) that \( \delta\Phi_\pm(r) \) is generally peaked (strongly so for large \( l \)-values) at the inner and outer surfaces of the step-function density profile at \( r = r_b^- \) and \( r = r_b^+ \), respectively. Typical numerical results are illustrated in Fig. 11, where \( Re \delta\Phi_\pm(r) \) and \( Im \delta\Phi_\pm(r) \) are plotted versus radius \( r \) for azimuthal mode number \( l = 3 \), filling factor \( \Delta = 0.1 \) and geometric factors \( r_b^+/r_w = 0.7 \) and \( r_b^+/r_w = 0.5 \). This corresponds (approximately) to the maximum-growth-rate parameters for the unstable \( l = 3 \) mode in Fig. 4. Without loss of generality, we pick the phase of the eigenfunction \( \delta\Phi(r) \) for mode number \( l = 3 \) so that \( \delta\Phi_- \) is purely real, and plot \( Re \delta\Phi_\pm(r) \) and \( Im \delta\Phi_\pm(r) \), normalized in units of \( \delta\Phi_- \), versus radius \( r \) in Fig. 11. Because \( \omega = \omega_\pm + i\gamma = 6.7/\omega_\pm + i1.4/\omega_\pm \) has nonzero real and imaginary parts determined from Eq. (25), it follows from Eqs. (21) and (22) that \( \delta\Phi_+ = (2.57 + i0.074)\delta\Phi_- \) is also complex. As expected from
Eq. (16), the eigenfunction $\Phi_j(r)$, plotted versus radius $r$ in Fig. 11, develops both real and imaginary components in the interval $r_b^- < r < r_w$.

As a final point, the right-hand side of Eq. (15) is equal to $4\pi e \delta n_l(r)$, where $\delta n_l(r)$ is the perturbed density of electrons. For the step-function equilibrium density profile in Eq. (7), it is clear from Eq. (15) that the perturbed charge density is equal to zero everywhere except at the surfaces $r = r_b^-$ and $r = r_b^+$. Where there are large (singular) perturbations in surface charge density, $\sigma_l(r_b^+) = (2\pi r_b^-)^{-1} (-e) \int_{r_b^-}^{r_b^+} (1+\epsilon) d \ln r \delta n_l(r)$ and $\sigma_l(r_b^-) = (2\pi r_b^+)^{-1} (-e) \int_{r_b^+}^{r_b^-} (1+\epsilon) d \ln r \delta n_l(r)$, where $\epsilon \to 0_+$. Without present algebraic details, which make use of Eqs. (21), (22) and (24), it can be shown that $\text{Re}[r_b^- \sigma_l(r_b^-)] + \text{Re}[r_b^+ \sigma_l(r_b^+)] = 0$ and $\text{Im}[r_b^- \sigma_l(r_b^-)] + \text{Im}[r_b^+ \sigma_l(r_b^+)] = 0$, which correspond to zero net perturbed charge density, i.e., $\int_0^{r_w} dr \delta n_l(r) = 0$, as expected.

**B. Continuously-varying density profile**

As a second example, we consider the continuously-varying density profile in Eq. (9) and Fig. 2. Here, $n_e^0(r)$ varies smoothly over the interval $0 < r < r_b^-$, and has an inverted population for sufficiently small values of $\Delta < 1/2$. Substituting Eq. (9) into Eq. (2) and integrating with respect to $r$ gives the angular velocity profile

$$\omega_E(r) = \hat{\omega}_d \left[ \Delta \frac{1}{2} \left( \frac{1}{2} - \Delta \right) - \frac{1}{3} \left( 2 - \Delta \right) \frac{r^4}{r_b^4} + \frac{1}{2} \frac{r^2}{r_b^2} \right]$$

for $0 < r < r_b^-$. 

---

**FIG. 10.** Stability–instability contour plots of $\hat{E}^2 = \hat{c}$ obtained from Eqs. (26) and (27) in the parameter space $(r_b^- / r_w, r_b^+ / r_w)$ for azimuthal mode number $l = 3$, and filling factors (a) $\Delta = 0.1$, (b) $\Delta = 0.2$, (c) $\Delta = 0.3$ and (d) $\Delta = 0.6$. 

where \( \hat{\omega}_d = \omega_{pe}^2/2 \omega_{ce} = 2 \pi \hat{n}_e e c / B_0 \). Here, \( \hat{\omega}_d \) can be eliminated in favor of \( N_e \) and the \( l = 1 \) diocotron frequency \( \omega_1 \) by means of Eqs. (10) and (11). This gives

\[
\hat{\omega}_d = \frac{r_b^2 / r_b^2}{(\Delta / 3 + 1/12)} \omega_1. \tag{31}
\]

Plots of the normalized profiles for \( \pi r_b^2 n_e^0(r)/N_e \) and \( \omega_{ce}(r)/\omega_1 \) versus \( r/r_b \) calculated from Eqs. (9) and (30), respectively, are shown in Fig. 12 for several values of the dimensionless parameter \( \Delta \) to illustrate the sensitive dependence of profile shape on \( \Delta \). Here, use has been made of Eqs. (10) and (31) to eliminate \( \hat{n}_e \) and \( \hat{\omega}_d \) in favor of \( N_e \) and \( \omega_1 \).

For \( n_e^0(r) \) specified by Eq. (9) and sufficiently small \( \Delta \), the shear in the angular velocity profile \( \omega_{ce}(r) \) defined in Eq. (30) is sufficiently large to drive the diocotron instability. For the equilibrium profiles in Eqs. (9) and (30), exact analytical solutions to the eigenvalue equation (1) are not tractable as was the case for the step-function density profile treated in Sec. III A. In the subsequent analysis of Eq. (1), we make use of the numerical code developed by White\(^{21} \) for solving eigenvalue equations in planar geometry. In this regard, it is convenient to introduce the stretched radial variable \( X \) defined by

\[
X = \ln \left( \frac{r}{r_b} \right), \quad \text{or} \quad \frac{r}{r_b} = \exp(X), \tag{32}
\]

so that \( r = 0 \) corresponds to \( X = -\infty \), \( r = r_b \) corresponds to \( X = 0 \) and \( r = r_w \) corresponds to \( X = X_w = \ln(r_w/r_b) \). Some straightforward algebra that makes use of Eqs. (1), (32) and \( \partial \hat{\omega}_d/\partial \hat{r} = \partial \hat{\omega}/\partial X \) gives the transformed eigenvalue equation.
\[
\left[ \frac{\partial^2}{\partial X^2} + F_I(X, \omega) \right] \delta \Phi_l(X) = 0. \tag{33}
\]

Here, \(F_I(X, \omega)\) is defined by

\[
F_I(X, \omega) = -l^2 + \frac{l}{\omega - l \omega F(X)} \frac{\partial}{\partial X} \frac{\omega^2_{pe}(X)}{\omega_c}. \tag{34}
\]

where \(\omega^2_{pe}(X)\) and \(\omega_F(X)\) are defined in Eqs. (9) and (30) with \(r/r_w = \exp(X)\). The transformed eigenvalue equation (33) is solved in the two regions corresponding to the following: the plasma interior (Region I, where \(0 < r < r_b\) or equivalently, \(0 < X < 0\)), and the vacuum region (Region II, where \(r_b < r < r_w\), or equivalently \(0 < X < X_w\) = \(\ln(r_w/r_b)\)). Requiring that \(\delta \Phi_l\) be regular at the origin and vanish at the conducting wall gives the boundary conditions

\[
\delta \Phi_l(X = -\infty) = 0, \tag{35}
\]

\[
\left[ \frac{\partial}{\partial X} \delta \Phi_l \right]_{X = -\infty} = 0,
\]

and

\[
\delta \Phi_l(X = X_w) = 0. \tag{36}
\]

In addition, for the continuously-varying density profile in Eq. (9), it is readily shown from Eqs. (33) and (34) that both \(\delta \Phi_l(X)\) and \(\partial \delta \Phi_l/\partial X\) are continuous across the surface of the plasma column at \(X = 0\) (corresponding to \(r = r_b\)), i.e.,

\[
\delta \Phi_l(X = 0) = \delta \Phi_l(X = 0), \tag{37}
\]

\[
\left[ \frac{\partial}{\partial X} \delta \Phi_l \right]_{X = 0} = \left[ \frac{\partial}{\partial X} \delta \Phi_l \right]_{X = -\infty}.
\]

In the vacuum region, where \(\omega^2_{pe}(X) = 0\), Eq. (33) reduces to

\[
\left[ \frac{\partial^2}{\partial X^2} + l^2 \right] \delta \Phi_l(X) = 0, \quad 0 < X < X_w, \tag{38}
\]

where \(X_w = \ln(r_w/r_b)\). Integrating Eq. (38), and enforcing \(\delta \Phi_l(X = X_w) = 0\) gives the solution

\[
\delta \Phi_l(X) = \delta \Phi_b \frac{\exp(-l(X - X_w)) - \exp(l(X - X_w))}{\exp(lX_w) - \exp(-lX_w)}.
\]

\[
0 < X < X_w. \tag{39}
\]

Here, the constant \(\delta \Phi_b = \delta \Phi_l(X = 0)\) is the perturbed potential at the surface of the plasma column \((X = 0)\).

Interior to the plasma, where \(\omega^2_{pe}(X)\) and \(\omega_F(X)\) are specified by Eqs. (9) and (30), the eigenvalue equation (33) reduces to

\[
\left[ \frac{\partial^2}{\partial X^2} + F_I(X, \omega) \right] \delta \Phi_l(X) = 0, \quad -\infty < X < 0. \tag{40}
\]

Here, \(F_I(X, \omega)\) is defined by

\[
F_I(X, \omega) = -l^2 + 2 \omega d \frac{\partial}{\partial X} \left\{ \left[ \Delta + \exp(2X) \right] \left[ 1 - \exp(2X) \right]^2 \right\}
\]

\[
\times \left\{ \omega - l \omega d \left( \Delta + \left( \frac{1}{2} - \Delta \right) \exp(2X) \right) \right. \\
\left. - \frac{1}{2} (2 - \Delta) \exp(4X) + \frac{1}{2} \exp(6X) \right\}^{-1}. \tag{41}
\]

In the present analysis, Eq. (40) is integrated numerically in Region I \((-\infty < X < 0)\) subject to the boundary conditions in Eq. (35) at \(X = -\infty\), and the solution for \(\delta \Phi_l(X)\) is matched at \(X = 0\) to the solution for \(\delta \Phi_l(X)\) in Region II given in Eq. (39) by imposing the boundary conditions in Eq. (37). For specified values of the dimensionless parameter \(\Delta\), total amount of plasma \((\omega_1)\) and location of the conducting wall \((r_b/r_w)\), this procedure gives numerical solutions for the eigenfunction \(\delta \Phi_l(X)\) and (complex) eigenfrequency \(\omega\).

In the subsequent analysis, it should be recognized that there are significant differences between the continuously-varying density profile in Eq. (9) and the step-function density profile in Eq. (7). First, the continuously-varying density profile in Eq. (9) is very sensitive to small increases in the dimensionless parameter \(\Delta\). This is evident from Fig. 12, where \(\Delta\) is varied from \(\Delta = 0\) to \(\Delta = 0.2\) at fixed values of \(N_c\) and \(r_b/r_w\). Second, the steep density gradient at the inner surface \(r = r_b\) in Eq. (7) (see also Fig. 1) tends to produce a strong version of the diocotron instability with sizeable growth rate \(Im \omega\) measured in units of \(\omega_1\) (see Figs. 3–7). By contrast, the density gradients in Eq. (9) and Fig. 12 are gentle, and we find in the subsequent analysis that the growth rates of the diocotron instability are correspondingly small with \(Im \omega \ll \omega_1\). Indeed, denoting the real oscillation frequency by \(Re \omega = \omega_r\), it is found that a small class of resonant particles located at radius \(r = r_s\) determined from the resonance condition

\[
\omega_r - l \omega_F(r_s) = 0 \tag{42}
\]

play a controlling role in determining properties of the diocotron instability for the continuously-varying density profile in Eq. (9). By contrast, for the case of the step-function density profile in Eq. (7), the diocotron instability calculated from the dispersion relation (25), which is a quadratic equation for \(\omega\), tends to be algebraic in nature.

Typical results obtained by numerically integrating the eigenvalue equation (40) and matching boundary conditions at \(X = 0\) as described earlier in this section are presented in Figs. 13 and 14 for the case where \(r_b/r_w = 0.5\). Shown in Fig. 13 are plots versus \(\Delta\) of the normalized real oscillation frequency \(Re \omega / \omega_1\) and growth rate \(Im \omega / \omega_1\) of the unstable diocotron modes with azimuthal mode numbers \(l = 2, 3, 4\). It is evident from Fig. 13 that the \(l = 3\) mode has the largest growth rate, and that the \(l = 1\) mode is stable with \(Im \omega = 0\) and \(Re \omega = \omega_r\), as expected. Moreover, from Fig. 13(b), the instability growth rate is strongest when \(\Delta = 0\), i.e., when the density depression in Fig. 12 is largest, and the instability growth rates decrease to negligibly small levels as \(\Delta\) is increased to modest values \((\Delta \leq 0.08)\). Similar behavior is evident in Fig. 14 where \(Re \omega / \omega_1\) and \(Im \omega / \omega_1\) are plotted versus the azimuthal mode number \(l\) for \(l = 1, 2, \ldots, 7\), and values of \(\Delta\) corresponding to \(\Delta = 0\), 0.01 and 0.03. With regard to the linear dependence of \(Re \omega\) on mode number \(l\) evident from the numerical results in Fig. 14(a), a remarkably good fit is provided by the empirical formula \(Re \omega = (5l - 4) \omega_1\). Finally, from Fig. 14(b), the \(l = 3\) mode exhibits the strongest instability, and the maximum growth rate decreases rapidly as \(\Delta\) is increased to small nonzero values.
Typical numerical results obtained for the radial dependence of the complex eigenfunction are illustrated in Figs. 15 and 16 for the choice of system parameters $\Delta = 0$ and $r_b/r_w = 0.5$ (see Fig. 14 for the corresponding values of $Re \omega$ and $Im \omega$). The most natural (but perhaps least informative) representation of the eigenfunction is in terms of the perturbed electrostatic potential, which is shown in Fig. 15 for azimuthal mode number $l = 2$. As expected, $\delta \Phi_l(r)$ has both real and imaginary parts in Fig. 15, and the eigenfunction has a broad radial structure with maximum magnitude where the plasma density is large. Because the eigenvalue equation (1) is homogeneous in the complex eigenfunction $\delta \Phi_l(r)$, it should be noted that $\delta \Phi_l(r)$ can be scaled by a factor $\exp(i\alpha)$, where $\alpha$ is a constant phase factor. In Fig. 15, when integrating Eq. (1) [or equivalently, Eq. (33)], we have chosen the phase $\alpha$ so that the eigenfunction $\delta \Phi_l(r)$ is purely real for small values of $r$ near the origin. A careful examination of Eq. (1) for small $r$ then shows that $[r^{-1}(\delta \rho/\delta r)(r \delta \Phi_l/\delta r) - l^2/r^2]Im \delta \Phi_l(r)$ is proportional to $[r^{-1}\delta \omega_p/\delta r]_{r = r_p} Re \delta \Phi_l(r)$. Therefore, as evident from Fig. 15, $Im \delta \Phi_l(r)$ vanishes until $r$ increases to $r_s$, the resonant radius that solves $\omega - l \omega_p(r_s)$.

As shown in Fig. 16, for mode numbers $l = 2, 3$ and 4, it is much more informative to plot the real and imaginary parts of the eigenfunction for the perturbed density $\delta n_l(r)$ and $Im \delta n_l(r)$ are very strongly peaked in a narrow radial region of the positive density gradient. Indeed, from the numerical solutions for $Re \omega = \omega_r$, it is found that the precise radial location $r = r_p$ of the localized density perturbation in Fig. 16 is determined from the resonance condition $\omega_r - l \omega_p(r_s)$.
where $\omega_E(r)$ is the angular velocity profile defined in Eq. (30). (Here, $\Delta=0$ and $r_b/r_w=0.5$ is assumed.

Note also from Fig. 16 that as $l$ increases from $l=2$ to $l=4$, the resonant radius $r=r_s$ moves progressively outwards towards the density maximum at $r_{\max}/r_b=(1/3)^{1/2}=0.577$ (for $\Delta=0$). This is further illustrated in Fig. 17 where the values of $r_s$ calculated numerically from Eqs. (30) and (42) and the numerical solutions for $\omega_\ell$ are plotted for mode numbers $l=1,2,\ldots,10$. In Fig. 17, the values of $r_s$ to the right of the density maximum but to the left of $r_b$ correspond to purely oscillatory modes with $\text{Im} \omega_\ell=0$ and mode numbers $l=6,\ldots,10$. For values of $r_s$ in Fig. 17 to the left of the density maximum, the $l=1$ mode, of course, is a stable oscillation with $\text{Im} \omega_1=0$ and $\text{Re} \omega_1=\omega_1$, whereas the $l=2,3,\ldots$ modes are unstable, with largest growth rate for $l=3$, and a negligibly small growth rate for $l\geq 7$ (see Fig. 14). Finally, it should also be pointed out in Fig. 16 that the real and imaginary parts of the eigenfunction $\delta n_\ell(r)$ satisfy charge conservation, $\int_0^{r_b} dr' \delta n_\ell(r')=0$, as expected.

For the continuously-varying density profile specified by Eq. (9), it is evident from Figs. 13, 14 and 16, that the diocotron instability is both weak ($\text{Im} \omega \ll |\text{Re} \omega|$) and resonant. Therefore, to better understand semi-quantitative features of the instability, it is useful to summarize briefly a formulation of the resonant diocotron instability developed originally by Briggs et al. We denote $\omega_\ell=\text{Re} \omega_\ell$ and $\gamma=\text{Im} \omega_\ell$, and Taylor expand the effective dispersion relation $D(\omega_\ell+i\gamma)=0$ in Eq. (4) for $|\gamma/\omega_\ell|\ll 1$. This gives

$$D(\omega_\ell+i\gamma) = D_0(\omega_\ell) + i \left[ D_1(\omega_\ell) + \gamma \frac{\partial D_1(\omega_\ell)}{\partial \omega_\ell} \right] + \cdots = 0,$$

where $D_0(\omega_\ell)=\text{Re} D(\omega_\ell)$ and $D_1(\omega_\ell)=\text{Im} D(\omega_\ell)$. Setting real and imaginary parts in Eq. (43) separately equal to zero then gives

FIG. 16. Plots of $\text{Re} \delta n_\ell(r)$ and $\text{Im} \delta n_\ell(r)$ versus $r/r_b$ obtained numerically for $\Delta=0$ and $r_b/r_w=0.5$ for azimuthal mode numbers (a) $l=2$, (b) $l=3$ and (c) $l=4$. The continuously-varying density profile $n_0(r)$ in Eq. (9) is also plotted versus $r/r_b$ in the figures.
To evaluate $D_\omega(\omega_\gamma)$ and $D_\omega(\omega_\lambda)$, we make use of

$$D_\omega(\omega_\gamma) = \frac{\partial D_\omega(\omega_\gamma)}{\partial \omega_\gamma},$$

(44)

To evaluate $D_\lambda(\omega_\gamma)$ and $D_\lambda(\omega_\lambda)$, we make use of

$$\lim_{\gamma \to 0_+} \frac{1}{\omega_r - \omega_E(r) + i \gamma} = \frac{P}{\omega_r - \omega_E(r)} - i \pi \delta(\omega_r - \omega_E(r))$$

(45)

in Eq. (4), where $P$ denotes Cauchy principal value. Substituting Eq. (45) into Eq. (4) and taking the limit $\gamma \to 0_+$ then gives

$$D_\lambda(\omega_\gamma) = \int_0^{r_w} \! dr r \left[ \frac{\partial}{\partial r} \delta(\omega_r - \omega_E(r)) \right] \left[ \frac{\partial}{\partial r} \frac{\partial \Phi}{\partial r} \right]$$

$$- \frac{1}{r \omega_{ce}} \frac{P \partial \omega^2_{pe}}{\partial r} \left[ \frac{\partial \Phi}{\partial r} \right]^2,$$

(46)

and

$$D_\lambda(\omega_\lambda) = \frac{1}{\omega_{ce}} \int_0^{r_w} \! dr r \frac{\partial \omega^2_{pe}(r)}{\partial r} \left[ \frac{\partial \omega_r - \omega_E(r)}{\partial r} \right] \left[ \frac{\partial \Phi}{\partial r} \right]^2$$

$$- \frac{\pi}{\omega_{ce}} \left[ \frac{\partial \omega^2_{pe}(r)}{\partial r} \frac{\partial \Phi}{\partial r} \right]_{r=r_s},$$

(47)

where the resonant radius $r_s$ solves $\omega_r = \omega_E(r_s)$. Substituting Eqs. (46) and (47) into Eq. (44) then gives

$$0 = D_\lambda(\omega_\gamma) = \int_0^{r_w} \! dr r \left[ \frac{\partial}{\partial r} \delta(\omega_r - \omega_E(r)) \right] \left[ \frac{\partial}{\partial r} \frac{\partial \Phi}{\partial r} \right]^2$$

$$- \frac{1}{r \omega_{ce}} \frac{P \partial \omega^2_{pe}}{\partial r} \left[ \frac{\partial \Phi}{\partial r} \right]^2,$$

(48)

and

$$\gamma = \frac{\pi}{\omega_{ce}} \left[ \frac{\partial \omega^2_{pe}(r)}{\partial r} \frac{\partial \Phi}{\partial r} \right]_{r=r_s}$$

$$\times \left[ - \frac{1}{\omega_r - \omega_E(r_s)} \right] \frac{\partial \Phi}{\partial r} \left[ \frac{\partial \Phi}{\partial r} \right]_{r=r_s}^{-1}.$$

(49)

Equation (48) plays the role of a dispersion relation for the real oscillation frequency $\omega_r$, whereas Eq. (49) determines the growth rate $\gamma$ of the resonant diocotron instability. Of course neither Eq. (48) nor Eq. (49) provide information on the detailed functional form of the complex eigenfunction $\Phi_\gamma(r)$. Nonetheless, important qualitative features of the instability are evident. In particular, for the continuously-varying density profile in Eq. (9), the factor $[\cdots]^{-1}$ in Eq. (49) is positive, so that the positive density gradient $[\partial \omega^2_{pe}/\partial r]_{r=r_s}$ drives the resonant diocotron instability for values of the resonant radius $r_s$ to the left of the density maximum.

IV. CONCLUSIONS

In this paper, we have made use of the electrostatic eigenvalue equation (1) to determine the influence of density profile shape on the diocotron instability in a low-density $(\omega^2_{pe} < \omega^2_{ce})$ non-neutral plasma column confined by a uniform axial magnetic field $B_0\hat{z}$. The assumptions and theoretical model were described in Sec. II, and in Sec. III detailed stability results were presented for two classes of equilibrium density profiles $n^0_r(r)$ with inverted population as a function of radius $r$. The first (Sec. III A) corresponds to the step-function density profile in Eq. (7) (see Fig. 1), whereas the second (Sec. III B) corresponds to the continuously-varying density profile in Eq. (9) (see Figs. 2 and 12). In both cases, the dimensionless parameter $\Delta$ controls the degree of “hollowness” of the equilibrium density profile, with $\Delta = 0$ corresponding to $n^0_r(r=0) = 0$. Detailed stability properties were calculated in Sec. III for a wide range of system parameters, including values of $\Delta$, radial location $r_w$ of the conducting wall, azimuthal mode number $l$, etc. As a general remark, in both cases, it was shown that small increases in the “filling factor” $\Delta$ from the value $\Delta = 0$ can have a large effect on the growth rate and detailed properties of the instability. In addition, for the step-function density profile in Eq. (7), which has a steep density gradient at the inner layer surface ($r = r_w$), the instability tends to be algebraic in nature and have a large growth rate in the unstable region of parameter space (see Figs. 3–7). By contrast, for the continuously-varying density profile in Eq. (9), the instability is typically much weaker (see Figs. 13 and 14) and involves a narrow class of resonant particles at radius $r = r_s$ satisfying the resonance condition in Eq. (42). To help better understand the qualitative features of the weak resonant diocotron instability, further work is clearly needed.
cotron instability, an expression for the growth rate $\gamma = \text{Im} \omega$ was presented in Eq. (49), which relates the growth rate $\gamma$ directly to the density gradient $\frac{\partial \omega_{pe}^2(r)}{\partial r}$ at the resonant radius $r = r_s$.

In conclusion, to help motivate future experimental studies, the present analysis has quantified the sensitive dependence of the diocotron instability growth rate and mode structure on the shape of the equilibrium density profile $n_e(r)$. Detailed stability properties have been calculated for profiles ranging from a thin annulus to a continuously varying density profile with inverted population. It is hoped that this work will motivate future experimental studies, both to test the validity of the linear stability analysis, and to help guide the planning of experiments to preferentially excite certain modes ($l$-values) and follow their nonlinear evolution. In this regard, it is important to note that the perturbed density $\delta n_l(r)$, rather than the perturbed potential $\delta \phi_l(r)$, is a particularly sensitive diagnostic of the detailed mode structure.

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