

# Instability Driven by Wall Impedance in Intense Charged Particle Beams

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## Abstract

The linearized Vlasov-Maxwell equations are used to investigate properties of the wall-impedance-driven instability for a long charge bunch with flat-top density profile propagating through a cylindrical pipe with radius  $r_w$  and wall impedance  $\tilde{Z}(\omega)$ . The stability analysis is valid for general value of the normalized beam intensity  $s_b = \hat{\omega}_{pb}^2 / 2\gamma_b^2 \omega_{\beta\perp}^2$  in the interval  $0 < s_b < 1$ .

## INTRODUCTION AND MODEL

High energy ion accelerators, transport systems and storage rings [1] have a wide range of applications ranging from basic research in high energy and nuclear physics, to applications such as spallation neutron sources, heavy ion fusion, and nuclear waste transmutation. Considerable recent analytical progress has been made in applying the Vlasov-Maxwell equations to investigate the detailed equilibrium and stability properties of intense charged particle beams [1]. Building on these advances, the present analysis reexamines the classical wall-impedance-driven instability [2-5], making use of the linearized Vlasov-Maxwell equations [1] for perturbations about a Kapchinskij-Vladimirskij (KV) beam equilibrium [6, 7] with flattop density profile. Compared with previous work, the present analysis based on the Vlasov-Maxwell equations constitutes a much more general approach. In particular, it enables us to solve for the dispersion relations and mode structures for arbitrary azimuthal mode number  $\ell$  in the transverse direction [8].

To summarize, the present analysis considers a very long charge bunch (bunch length  $\ell_b \gg$  bunch radius  $r_b$ ) with directed axial kinetic energy  $(\gamma_b - 1)m_b c^2$  propagating in the  $z$ -direction through a cylindrical pipe with constant radius  $r_w$  and (complex) wall impedance  $\tilde{Z}(\omega)$ . The analysis is carried out in the smooth-focusing approximation, where the applied transverse focusing force is modeled by  $\mathbf{F}_{foc} = -\gamma_b m_b \omega_{\beta\perp}^2 \mathbf{x}_\perp$ . Here,  $\gamma_b = (1 - \beta_b^2)^{-1/2}$  is the relativistic mass factor,  $V_b = \beta_b c$  is the directed axial velocity of the charge bunch,  $m_b$  is the particle rest mass,  $\omega_{\beta\perp} = const.$  is the applied focusing frequency, and  $\mathbf{x}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$  is the transverse displacement of a beam particle from the cylinder axis. Denoting the number density of beam particles by  $\hat{n}_b$  and the particle charge

by  $e_b$ , it is convenient to introduce the relativistic plasma frequency  $\hat{\omega}_{pb}$  defined by  $\hat{\omega}_{pb} = (4\pi\hat{n}_b e_b^2 / \gamma_b m_b)^{1/2}$  and the normalized (dimensionless) beam intensity  $s_b$  defined by  $s_b = \hat{\omega}_{pb}^2 / 2\gamma_b^2 \omega_{\beta\perp}^2$  [1].

An important feature of the analysis is that it is carried out for arbitrary value of  $s_b$  in the interval  $0 < s_b < 1$ , assuming perturbations about a KV equilibrium with flat-top density profile. Illustrative parameters for intense beam systems range from the very small value  $s_b = 1.36 \times 10^{-4}$  in the Tevatron, where the particles are highly relativistic, to the intermediate value  $s_b = 0.08$  in the low-energy, moderate-intensity Proton Storage Ring (PSR) experiment, to  $s_b \simeq 0.98$  in the low-emittance, space-charge-dominated beams for heavy ion fusion. Finally, the present analysis considers the case where the axial momentum spread is negligibly small. Furthermore, the functional form of the wall impedance  $\tilde{Z}(\omega)$  is not specified, although the case of small impedance ( $|\tilde{Z}| \ll 1$ ) is considered.

To describe stability properties of the charge bunch, we make use of a kinetic description based on the Vlasov-Maxwell equations [1]. For simplicity, the analysis considers small-amplitude perturbations about the axisymmetric, axially uniform, quasi-steady-state equilibrium distribution function [6, 7]

$$f_b^0(r, \mathbf{p}_\perp) = \frac{\hat{n}_b}{2\pi\gamma_b m_b} \delta(H_\perp - \hat{T}_{\perp b}) \delta(p_z - \gamma_b m_b \beta_b c). \quad (1)$$

In Eq. (1),  $\hat{n}_b$  and  $\hat{T}_{\perp b}$  are positive constants, and  $H_\perp$  is the transverse Hamiltonian defined by

$$H_\perp = \frac{1}{2\gamma_b m_b} p_\perp^2 + \frac{1}{2} \gamma_b m_b \omega_{\beta\perp}^2 r^2 + e_b [\phi^0(r) - \beta_b A_z^0(r)], \quad (2)$$

where  $r = (x^2 + y^2)^{1/2}$  is the radial distance from the cylinder axis, and  $p_\perp = (p_x^2 + p_y^2)^{1/2}$  is the transverse momentum. In Eq. (2), the equilibrium self-field potentials are determined self-consistently in terms of  $f_b^0(r, \mathbf{p})$  from the steady-state Maxwell equations. Note that the distribution function in Eq. (1) is *cold* in the axial direction. An attractive feature of the choice of  $f_b^0(r, \mathbf{p})$  in Eq. (1) is that the corresponding equilibrium number density,  $n_b^0(r) = \int d^3 p f_b^0(r, \mathbf{p})$ , has the flattop profile [6, 7]

$$n_b^0(r) = \begin{cases} \hat{n}_b = const., & 0 \leq r < r_b, \\ 0, & r_b < r \leq r_w. \end{cases} \quad (3)$$

Here,  $\hat{n}_b = const.$  is the number density of beam particles,

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and the edge radius  $r_b$  is determined self-consistently from

$$\frac{2\hat{T}_{\perp b}}{\gamma_b m_b} = \left( \omega_{\beta\perp}^2 - \frac{1}{2\gamma_b^2} \hat{\omega}_{pb}^2 \right) r_b^2 \equiv \nu_b^2 r_b^2. \quad (4)$$

In Eq. (4), we have introduced the quantity  $\nu_b^2$  defined by

$$\nu_b^2 = \omega_{\beta\perp}^2 - \frac{1}{2\gamma_b^2} \hat{\omega}_{pb}^2 = \omega_{\beta\perp}^2 (1 - s_b), \quad (5)$$

where  $s_b = \hat{\omega}_{pb}^2 / 2\gamma_b^2 \omega_{\beta\perp}^2$  is a dimensionless measure of the normalized beam intensity. Note from Eq. (5) that  $\nu_b = \omega_{\beta\perp} (1 - s_b)^{1/2}$  corresponds to the (depressed) betatron frequency for transverse particle oscillations in the equilibrium field configuration.

## LINEARIZED EQUATIONS

To investigate the instability, we express  $f_b(\mathbf{x}, \mathbf{p}, t) = f_b^0(r, \mathbf{p}) + \delta f_b(\mathbf{x}, \mathbf{p}, t)$ , and make use of the linearized Vlasov-Maxwell equations [1, 8] to determine the self-consistent evolution of  $\delta f_b(\mathbf{x}, \mathbf{p}, t)$ ,  $\delta \mathbf{E}^s(\mathbf{x}, t)$  and  $\delta \mathbf{B}^s(\mathbf{x}, t)$  for small-amplitude perturbations. The perturbed fields are expressed as  $\delta \mathbf{E}^s = -\nabla \delta \phi - c^{-1} \partial \delta \mathbf{A} / \partial t$  and  $\delta \mathbf{B}^s = \nabla \times \delta \mathbf{A}$ , and use is made of the Lorentz gauge condition,  $\nabla \cdot \delta \mathbf{A} = -c^{-1} \partial \delta \phi / \partial t$ , to relate  $\delta \mathbf{A}$  and  $\delta \phi$ . The linearized Vlasov-Maxwell equations are analyzed [8] for perturbations of the form

$$\delta \psi(\mathbf{x}, t) = \delta \psi^\ell(r) \exp(i\ell\theta + ik_z z - i\omega t), \quad (6)$$

where  $\ell = 1, 2, \dots$  is the azimuthal mode number of the perturbation,  $k_z$  is the axial wavenumber, and  $\omega$  is the oscillation frequency. We consider perturbations with sufficiently low frequency and long axial wavelength that

$$\frac{|\omega| r_b}{c} \ll 1, \quad |k_z| r_b \ll 1, \quad \text{and} \quad \left| k_z^2 - \frac{\omega^2}{c^2} \right| r_w^2 \ll 1. \quad (7)$$

The linearized Vlasov-Maxwell equations can be simplified within the context of the inequalities in Eq. (7). Without presenting algebraic details [1, 8], it follows that the perturbed transverse force  $\delta \mathbf{F}_\perp$  on a beam particle can be approximated by

$$\delta \mathbf{F}_\perp = -e_b \nabla_\perp \left( \delta \phi - \frac{1}{c} v_z \delta A_z \right). \quad (8)$$

Similarly, for the low-frequency, long-wavelength perturbations consistent with Eq. (7), it can be shown that the perturbed longitudinal force can be neglected [7, 8]. Moreover, because the axial momentum spread is negligibly small for the distribution function in Eq. (1), we approximate  $\int d^3 p v_z \delta f_b = \beta_b c \int d^3 p \delta f_b$ .

In summary, making use of the approximations outlined above, the linearized Vlasov-Maxwell equations can be approximated by [8]

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - \gamma_b m_b \nu_b^2 \mathbf{x}_\perp \cdot \frac{\partial}{\partial \mathbf{p}} \right) \delta f_b \\ & = \frac{e_b}{\gamma_b m_b} \mathbf{p}_\perp \cdot \nabla_\perp \left( \delta \phi - \frac{1}{c} v_z \delta A_z \right) \frac{\partial f_b^0}{\partial H_\perp}, \quad (9) \end{aligned}$$

where  $\delta \phi$  and  $\delta A_z$  are determined from

$$\nabla_\perp^2 \delta \phi = -4\pi e_b \delta n_b, \quad (10)$$

$$\nabla_\perp^2 \delta A_z = -4\pi e_b \beta_b \delta n_b. \quad (11)$$

Here,  $\nabla_\perp^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ ,  $V_b = \beta_b c$  is the average axial velocity, and the perturbed number density is defined by  $\delta n_b = \int d^3 p \delta f_b$ .

Equations (9)–(11) are to be solved in the beam interior ( $0 \leq r < r_b$ ) and in the vacuum region ( $r_b < r \leq r_w$ ) outside the beam, enforcing the appropriate boundary conditions at the conducting wall located at radius  $r = r_w$ . For present purposes, we describe the wall impedance by a complex scalar function,  $\tilde{Z}(\omega) = \tilde{Z}_r + i\tilde{Z}_i$ , where  $\omega$  is the oscillation frequency in Eq. (6). The boundary condition on the perturbed tangential electric and magnetic fields at  $r = r_w^- \equiv [r_w(1 - \epsilon)]_{\epsilon \rightarrow 0^+}$  can be expressed as [2-5, 8]

$$[\delta \mathbf{E}_t]_{r_w^-} = \tilde{Z}(\omega) \hat{\mathbf{n}} \times [\delta \mathbf{B}_t]_{r_w^-}. \quad (12)$$

Here,  $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_r$  is a unit vector pointing outward from the cylindrical conducting wall surface. In what follows we assume that the metal wall is almost perfectly conducting, implying that  $|\tilde{Z}(\omega)| \ll 1$ . Making use of  $(\nabla \times \delta \mathbf{B})_r = c^{-1} \partial \delta E_r / \partial t$  in the vacuum region, the boundary conditions in Eq. (12) can be expressed for  $|\tilde{Z}| \ll 1$  as

$$\begin{aligned} k_z [\delta \phi^\ell]_{r_w^-} - \frac{\omega}{c} [\delta A_z^\ell]_{r_w^-} &= i\tilde{Z} \left[ \frac{\partial}{\partial r} \delta A_z^\ell \right]_{r_w^-}, \quad (13) \\ \frac{\ell}{r_w} [\delta \phi^\ell]_{r_w^-} &= -i\tilde{Z} \left\{ \frac{\omega}{c} \left[ \frac{\partial}{\partial r} \delta \phi^\ell \right]_{r_w^-} + k_z \left[ \frac{\partial}{\partial r} \delta A_z^\ell \right]_{r_w^-} \right\}. \end{aligned}$$

Equation (13) expresses the boundary conditions at the conducting wall in terms of the impedance  $\tilde{Z}(\omega)$  and the perturbed potentials,  $\delta \phi$  and  $\delta A_z$ . In the limit of zero impedance,  $\tilde{Z} \rightarrow 0$ , Eq. (13) reduces to  $[\delta \phi^\ell]_{r_w^-} = 0 = [\delta A_z^\ell]_{r_w^-}$ , corresponding to the boundary conditions expected for a perfectly conducting, cylindrical wall. Depending on the frequency regime, there are several models of wall impedance  $\tilde{Z}(\omega)$  that can be used in the boundary conditions in Eq. (13) [4].

## STABILITY ANALYSIS

In the analysis of Eqs. (9)–(11), we introduce the new independent variables  $\tau$  and  $Z$  defined by  $\tau = t - z/V_b$  and  $Z = z$ . The perturbation in Eq. (6) can be expressed as

$$\delta \psi(\mathbf{x}, Z, \tau) = \delta \psi^\ell(r) \exp[i\ell\theta - i\omega\tau - i(\Omega/V_b)Z], \quad (14)$$

where  $\ell = 1, 2, \dots$ , is the azimuthal mode number,  $\omega$  is the oscillation frequency, and

$$\Omega/V_b = (\omega - k_z V_b)/V_b \quad (15)$$

is the effective axial wavenumber of the perturbation in the new variables  $(Z, \tau)$ . If the charge bunch experiences a perturbation for  $\tau > 0$  with real oscillation frequency  $\omega$ , it

is evident from Eqs. (14) and (15) that  $\Omega/V_b$  represents the spatial oscillation and growth (or damping) of the perturbation as a function of axial position  $Z$ .

Assuming perturbations of the form in Eq. (14) for  $Im\Omega > 0$  and integrating Eq. (9), it is found that a class of solutions exists with density perturbation amplitude  $\delta n_b^\ell(r) = \int d^2p \delta f_b^\ell(r, \mathbf{p}_\perp)$  localized at the surface of the charge bunch ( $r = r_b$ ). Without presenting algebraic details [8], we obtain

$$4\pi e_b \delta n_b^\ell(r) = -\frac{2\ell}{r_b} \chi_b^\ell(\Omega) [\delta\phi^\ell(r) - \beta_b \delta A_z^\ell(r)] \delta(r - r_b). \quad (16)$$

Here, the response function  $\chi_b^\ell(\Omega)$  is defined by

$$\chi_b^\ell(\Omega) = -\frac{\hat{\omega}_{pb}^2}{2\ell 2^\ell \nu_b^2} \sum_{m=0}^{\ell} \frac{\ell!}{m!(\ell-m)!} \frac{(\ell-2m)\nu_b}{\Omega - (\ell-2m)\nu_b}, \quad (17)$$

where  $\Omega = \omega - k_z V_b$  is the Doppler shifted frequency,  $\hat{\omega}_{pb} = (4\pi \hat{n}_b e_b^2 / \gamma_b m_b)^{1/2}$  is the relativistic plasma frequency, and  $\nu_b = (\omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2 / 2\gamma_b^2)^{1/2}$  is the depressed betatron frequency. As expected, the response function in Eq. (17) has a rich harmonic content at harmonics of  $\nu_b$ . We define  $\delta\psi^\ell(r) = \delta\phi^\ell(r) - \beta_b \delta A_z^\ell(r)$ , and denote  $\delta\hat{\psi}^\ell = \delta\psi^\ell(r_b)$ ,  $\delta\hat{\phi}^\ell = \delta\phi^\ell(r_b)$  and  $\delta\hat{A}_z^\ell = \delta A_z^\ell(r_b)$ . Substituting Eq. (16), Maxwell's equations (10) and (11) become [8]

$$\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\ell^2}{r^2} \right) \delta\phi^\ell(r) = \frac{2\ell}{r_b} \chi_b^\ell(\Omega) \delta\hat{\psi}^\ell \delta(r - r_b), \quad (18)$$

$$\left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\ell^2}{r^2} \right) \delta A_z^\ell(r) = \frac{2\ell}{r_b} \beta_b \chi_b^\ell(\Omega) \delta\hat{\psi}^\ell \delta(r - r_b), \quad (19)$$

for azimuthal mode numbers  $\ell = 1, 2, \dots$ .

Equations (18) and (19), derived for perturbations about the equilibrium distribution in Eq. (1) with flattop-density profile, constitute the final forms of the eigenvalue equations used in the present stability analysis. Here, Eqs. (18) and (19) are to be solved over the interval  $0 \leq r \leq r_w$  for the eigenfunctions  $\delta\phi^\ell(r)$  and  $\delta A_z^\ell(r)$  and eigenvalue  $\Omega$ , subject to the condition that  $\delta\phi^\ell(r)$  and  $\delta A_z^\ell(r)$  be regular at the origin ( $r = 0$ ), and satisfy the boundary conditions in Eq. (13) at the conducting wall ( $r = r_w$ ).

Equations (18) and (19) can be solved exactly for the eigenfunctions  $\delta\phi^\ell(r)$  and  $\delta A_z^\ell(r)$  in the interval  $0 \leq r \leq r_w$ , and the boundary condition (13) enforced at the conducting wall [8]. We introduce

$$\begin{aligned} \Delta' &= -2 \frac{\omega r_w}{\ell c} \left( 1 + \frac{k_z V_b}{\omega} \right) i \tilde{Z}(\omega), \\ \Delta &= -2 \frac{\ell c}{\omega r_w} \left[ 1 + \frac{k_z^2 r_w^2}{\ell^2} \left( 1 + \frac{\omega}{k_z V_b} \right) \right] i \tilde{Z}(\omega), \end{aligned} \quad (20)$$

where  $\Delta'$  and  $\Delta$  are treated as small parameters with  $|\Delta'|, |\Delta| \ll 1$ . Some straightforward algebra [8] then leads to

the dispersion relation

$$\begin{aligned} D_b^\ell(\Omega) &= 1 + \frac{1}{\gamma_b^2} \left[ 1 - \left( \frac{r_b}{r_w} \right)^{2\ell} \right] \chi_b^\ell(\Omega) \\ &+ \left( \frac{r_b}{r_w} \right)^{2\ell} \chi_b^\ell(\Omega) [\beta_b^2 \Delta - \Delta'] = 0. \end{aligned} \quad (21)$$

Equation (21) is the final form of the dispersion relation derived from the linearized Vlasov-Maxwell equations for perturbations about the equilibrium distribution function in Eq. (1) with corresponding flattop density profile in Eq. (3). The dispersion relation (21) is valid for low-frequency long-wavelength perturbations consistent with Eq. (7), and can be applied over a wide range of normalized beam intensity in the range  $0 < s_b = \hat{\omega}_{pb}^2 / 2\gamma_b^2 \omega_{\beta\perp}^2 < 1$ .

The dispersion relation (21) can be used to investigate detailed stability properties for azimuthal mode numbers  $\ell = 1, 2, 3, \dots$ . As an example, we consider dipole-mode perturbations with  $\ell = 1$ . In this case, it follows from Eq. (17) that the response function  $\chi_b^{\ell=1}(\Omega)$  is given by

$$\chi_b^{\ell=1}(\Omega) = -\frac{\hat{\omega}_{pb}^2/2}{\Omega^2 - \nu_b^2}, \quad (22)$$

where  $\nu_b^2 = \omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2 / 2\gamma_b^2$ . Substituting Eq. (22) into Eq. (21), the dispersion relation reduces to

$$\Omega^2 = \omega_{\beta\perp}^2 - \left( \frac{r_b^2}{r_w} \right)^2 \frac{\hat{\omega}_{pb}^2}{2\gamma_b^2} - \left( \frac{r_b}{r_w} \right)^2 \beta_b^2 \hat{\omega}_{pb}^2 \frac{c}{\omega r_w} i \tilde{Z}(\omega), \quad (23)$$

where use has been made of  $\nu_b^2 = \omega_{\beta\perp}^2 - \hat{\omega}_{pb}^2 / 2\gamma_b^2$ , and we have approximated  $\beta_b^2 \Delta - \Delta' = -(2i\beta_b^2 c \ell / \omega r_w) \tilde{Z}(\omega)$ .

The dipole-mode dispersion relation (23) is valid over the entire allowed range of normalized beam intensity ( $0 < s_b < 1$ ) and can be used to investigate detailed stability properties for a wide variety of choices of impedance function  $\tilde{Z}(\omega)$ . The application of Eq. (23) is discussed in more detail in Ref. 8.

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