MACROSCOPIC FLUID APPROACH TO THE COHERENT BEAM-BEAM INTERACTION

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Abstract

Building on the Radon transform of the Vlasov-Poisson equations, a macroscopic fluid model for the coherent beam-beam interaction has been developed. It is shown that the Vlasov equation, expressed in action-angle variables, can be reduced to a closed set of hydrodynamic (fluid) equations for the beam density and current velocity. The linearized one-dimensional equations have been analysed, and explicit expressions for the coherent beam-beam tuneshifts are presented.

1 INTRODUCTION

In a colliding-beam storage ring device, the evolution of each beam is strongly affected by the electromagnetic force produced by the counter-propagating beam. A basic feature of this coherent interaction is linear mode coupling, also known as the coherent beam-beam resonance.

The problem of coherent beam-beam resonances in one dimension (the vertical direction) was first studied by Chao and Ruth [1] by solving the linearized Vlasov-Poisson equations. They considered the simplest case of a symmetric collider and obtained explicit expressions for the resonance stopbands. The purpose of the present paper is to extend their results to the case of an asymmetric circular collider.

Based on the Radon transform [2, 3], a macroscopic fluid model of the coherent beam-beam interaction is developed. The linearized macroscopic fluid equations are then solved, and a generalized stability criterion for a coherent beam-beam resonance of arbitrary order is derived.

2 THE RADON TRANSFORM

We begin with the one-dimensional Vlasov-Poisson equations describing the nonlinear evolution of the beams in the vertical (y) direction

$$\frac{\partial f_k}{\partial \theta} + \nu_k p \frac{\partial f_k}{\partial y} - \frac{\partial \mathcal{H}_k}{\partial y}\frac{\partial f_k}{\partial p} = 0, \tag{2.1}$$

$$\mathcal{H}_k = \frac{\nu_k}{2}\left(p^2 + y^2\right) + \lambda_k \delta_p(\theta) V_k(y;\theta), \tag{2.2}$$

$$\frac{\partial^2 V_k}{\partial y^2} = 4\pi \int dp f_{3-k}(y,p;\theta), \tag{2.3}$$

$$\lambda_k = \frac{R r_e N_{3-k} \beta^*_{ky}}{\gamma_{k0} L_{(3-k)x}} \frac{1+\beta_{k0}\beta_{(3-k)0}}{\beta_{k0}^2} \simeq \frac{2 R r_e N_{3-k}\beta^*_{ky}}{\gamma_{k0} L_{(3-k)x}}. \tag{2.4}$$

Here, $(k = 1, 2)$ labels the beam, $f_k(y, p; \theta)$ is the distribution function, θ is the azimuthal angle, ν_k is the betatron tune in vertical direction, R is the mean machine radius, r_e is the classical electron radius, $N_{1,2}$ is the total number of particles in either beam, $V_k(y; \theta)$ is the normalized beam-beam potential, β^*_{ky} is the vertical beta-function at the interaction point, and L_{kx} is the horizontal dimension of the beam ribbon [1]. The one-dimensional Poisson equation (2.3) can be readily solved to give

$$V_k(y;\theta) = 2\pi \int dy' dp' f_{3-k}(y',p';\theta)|y-y'|. \tag{2.5}$$

Transforming to action-angle variables (J, φ), we rewrite Eqs. (2.1) and (2.2) in the form

$$\frac{\partial f_k}{\partial \theta} + \frac{\partial}{\partial \varphi}\left[\left(\nu_k + \lambda_k \delta_p(\theta)\frac{\partial V_k}{\partial J}\right) f_k\right] - \frac{\partial}{\partial J}\left(\lambda_k \delta_p(\theta)\frac{\partial V_k}{\partial \varphi} f_k\right) = 0, \tag{2.6}$$

$$\mathcal{H}_k = \nu_k J + \lambda_k \delta_p(\theta) V_k(\varphi, J; \theta), \tag{2.7}$$

where

$$V_k(\varphi, J;\theta) = 2\pi \int d\varphi' dJ' f_{3-k}(\varphi', J';\theta) \times \left|\sqrt{2J}\cos\varphi - \sqrt{2J'}\cos\varphi'\right|. \tag{2.8}$$

Next we perform the Radon transform defined as [2, 3]

$$f_k(\varphi, J;\theta) = \int d\xi \varrho_k(\varphi,\xi;\theta)\delta[J - v_k(\varphi,\xi;\theta)], \tag{2.9}$$

and obtain the hydrodynamic equations

$$\frac{\partial \varrho_k}{\partial \theta} + \frac{\partial}{\partial \varphi}\left[\left(\nu_k + \lambda_k \delta_p(\theta)\frac{\partial V_k}{\partial v_k}\right)\varrho_k\right] = 0, \tag{2.10}$$

$$\frac{\partial (\varrho_k v_k)}{\partial \theta} + \frac{\partial}{\partial \varphi}\left[\left(\nu_k + \lambda_k \delta_p(\theta)\frac{\partial V_k}{\partial v_k}\right)\varrho_k v_k\right] + \lambda_k \delta_p(\theta)\frac{\partial V_k}{\partial \varphi}\varrho_k = 0, \tag{2.11}$$

where ϱ_k is the Radon image of the distribution function f_k. The integration variable ξ is regarded as a Lagrange variable, that keeps track of the detailed information about the action J. It us usually determined by the condition that the distribution function f_k be equal to a specified distribution [3], from which $J = v_k(\varphi, \xi; \theta)$. Taking into account

Eq. (2.10), the beam density can be further eliminated from Eq. (2.11), which yields the result

$$\frac{\partial v_k}{\partial \theta} + \left(\nu_k + \lambda_k \delta_p(\theta) \frac{\partial V_k}{\partial v_k} \right) \frac{\partial v_k}{\partial \varphi} + \lambda_k \delta_p(\theta) \frac{\partial V_k}{\partial \varphi} = 0, \tag{2.12}$$

where

$$V_k(\varphi, v_k; \theta) = 2\sqrt{2}\pi \int d\varphi' d\xi' \varrho_{3-k}(\varphi', \xi'; \theta) \times \left| \sqrt{v_k(\varphi, \xi; \theta)} \cos\varphi - \sqrt{v_{3-k}(\varphi', \xi'; \theta)} \cos\varphi' \right|. \tag{2.13}$$

It is important to note that Eqs. (2.10) and (2.12) comprise a closed set, that is (as can be easily checked) equations for higher moments can be reduced to these two equations.

At this point we make the important conjecture that Eqs. (2.10) and (2.12) possess a stationary solution that is independent of the angle variable φ. Without loss of generality we choose

$$v_k^{(0)} = \xi = \text{const}, \qquad \varrho_k^{(0)} = G(\xi) = \text{const}. \tag{2.14}$$

3 SOLUTION OF THE LINEARIZED EQUATIONS

Expressing $\varrho_k = \varrho_k^{(0)} + \varrho_k^{(1)}$ and $v_k = v_k^{(0)} + v_k^{(1)}$, the linearized hydrodynamic equations can be written as

$$\frac{\partial \varrho_k^{(1)}}{\partial \theta} + \widetilde{\nu}_k \frac{\partial \varrho_k^{(1)}}{\partial \varphi} + \lambda_k \delta_p(\theta) \varrho_k^{(0)} \frac{\partial^2 V_k^{(1)}}{\partial \varphi \partial v_k} = 0, \tag{3.1}$$

$$\frac{\partial v_k^{(1)}}{\partial \theta} + \widetilde{\nu}_k \frac{\partial v_k^{(1)}}{\partial \varphi} + \lambda_k \delta_p(\theta) \frac{\partial V_k^{(1)}}{\partial \varphi} = 0. \tag{3.2}$$

Here $\widetilde{\nu}_k$ is the incoherently perturbed betatron tune, defined by

$$\widetilde{\nu}_k = \nu_k + \frac{\lambda_k}{2\pi} \left\langle \frac{\partial V_k^{(0)}}{\partial v_k} \right\rangle_{\varphi}, \tag{3.3}$$

where the angular bracket implies an average over the angle variable. Next we determine the derivatives of the first-order beam-beam potential $V_k^{(1)}$ entering the linearized hydrodynamic equations corresponding to

$$\frac{\partial V_k^{(1)}}{\partial \varphi} = -2\pi \sqrt{2\xi} \sin\varphi \int d\varphi' d\xi' \varrho_{3-k}^{(1)}(\varphi', \xi'; \theta) \times \text{sgn}\left(\sqrt{\xi} \cos\varphi - \sqrt{\xi'} \cos\varphi' \right), \tag{3.4}$$

$$\frac{\partial^2 V_k^{(1)}}{\partial \varphi \partial v_k} = -\pi \sqrt{\frac{2}{\xi}} \sin\varphi \int d\varphi' d\xi' \varrho_{3-k}^{(1)}(\varphi', \xi'; \theta) \times \text{sgn}\left(\sqrt{\xi} \cos\varphi - \sqrt{\xi'} \cos\varphi' \right). \tag{3.5}$$

Finally, we obtain the linearized equation for the beam density

$$\frac{\partial \varrho_k^{(1)}}{\partial \theta} + \widetilde{\nu}_k \frac{\partial \varrho_k^{(1)}}{\partial \varphi} - \pi \lambda_k \delta_p(\theta) \times \sqrt{\frac{2}{\xi}} \varrho_k^{(0)}(\xi) \sin\varphi \int d\varphi' d\xi' \varrho_{3-k}^{(1)}(\varphi', \xi'; \theta) \times \text{sgn}\left(\sqrt{\xi} \cos\varphi - \sqrt{\xi'} \cos\varphi' \right) = 0. \tag{3.6}$$

In order to solve Eq. (3.6), we note that the function $\varrho_k^{(1)}$ may be represented as

$$\varrho_k^{(1)}(\varphi, \xi; \theta) = \frac{\varrho_k^{(0)}(\xi)}{\sqrt{\xi}} \mathcal{R}_k(\varphi, \xi; \theta). \tag{3.7}$$

Assuming the function $G(\xi)$ in Eq. (2.14) to be of the form

$$G(\xi) = \frac{1}{2\pi\sigma_k^2} \exp\left(-\frac{\xi}{\sigma_k^2} \right) \tag{3.8}$$

for small vertical beam sizes σ_k, we obtain

$$\frac{\varrho_k^{(0)}(\xi) \varrho_{3-k}^{(0)}(\xi')}{\sqrt{\xi\xi'}} = \frac{\exp\left(-\frac{\xi}{\sigma_k^2} - \frac{\xi'}{\sigma_{3-k}^2} \right)}{(2\pi)^2 \sigma_k^2 \sigma_{3-k}^2 \sqrt{\xi\xi'}}$$

$$= \frac{\exp\left(-\frac{\xi'}{\sigma_{3-k}^2} + \frac{\xi'}{\sigma_k^2} - \frac{2\sqrt{\xi\xi'}}{\sigma_k^2} \right)}{(2\pi)^2 \sigma_k^2 \sigma_{3-k}^2 \sqrt{\xi\xi'}} \exp\left[-\frac{\left(\sqrt{\xi} - \sqrt{\xi'} \right)^2}{\sigma_k^2} \right]$$

$$\sim \sqrt{\pi} \sigma_k \frac{\varrho_k^{(0)}(\xi) \varrho_{3-k}^{(0)}(\xi')}{\sqrt{\xi\xi'}} \delta\left(\sqrt{\xi} - \sqrt{\xi'} \right). \tag{3.9}$$

If $\mathcal{R}_k$ does not depend on the Lagrange variable ξ, making use of Eq. (3.9), we rewrite Eq. (3.6) as

$$\frac{\partial \mathcal{R}_k}{\partial \theta} + \widetilde{\nu}_k \frac{\partial \mathcal{R}_k}{\partial \varphi} - \pi \widetilde{\lambda}_k \delta_p(\theta) \sin\varphi \times \int d\varphi' \mathcal{R}_{3-k}(\varphi'; \theta) \text{sgn}(\cos\varphi - \cos\varphi') = 0, \tag{3.10}$$

where

$$\widetilde{\lambda}_k = \sqrt{\frac{2}{\pi}} \lambda_k \frac{\sigma_k}{\sigma_{3-k}\Sigma}, \qquad \Sigma = \sqrt{\sigma_k^2 + \sigma_{3-k}^2}. \tag{3.11}$$

Note that this approximation is valid if and only if the perturbed betatron tunes in Eq. (3.3) do not depend on ξ, which in general is not the case. This leads to an effect similar to Landau damping, well-known in plasma physics, which we shall neglect in what follows. Fourier transforming Eq. (3.10) yields

$$\frac{\partial \widetilde{\mathcal{R}}_k(n)}{\partial \theta} + in\widetilde{\nu}_k \widetilde{\mathcal{R}}_k(n) - \frac{\widetilde{\lambda}_k}{2} \delta_p(\theta) \sum_{m=-\infty}^{\infty} \mathcal{M}_{nm} \widetilde{\mathcal{R}}_{3-k}(m) = 0, \tag{3.12}$$

where

$$\widetilde{\mathcal{R}}_k(n; \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \mathcal{R}_k(\varphi; \theta) \exp(-in\varphi), \tag{3.13}$$

$$\mathcal{M}_{nm} = \int\limits_0^{2\pi} d\varphi \int\limits_0^{2\pi} d\varphi' e^{-in\varphi} \sin\varphi e^{im\varphi'} \times \text{sgn}(\cos\varphi - \cos\varphi'). \qquad (3.14)$$

In order to determine the infinite matrix $\mathcal{M}$, we utilize the integral representation of the sign-function

$$\text{sgn}(x) = \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \exp(i\lambda x). \qquad (3.15)$$

As a result, we obtain

$$\mathcal{M}_{nm} = 4\pi n i^{n-m+1} \int\limits_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2} \mathcal{J}_m(\lambda) \mathcal{J}_n(\lambda)$$

$$= \begin{cases} -\frac{32in}{[(n+m)^2-1][(n-m)^2-1]}, & \text{for } n+m = \text{even}, \\ 0, & \text{for } n+m = \text{odd}, \end{cases} \qquad (3.16)$$

$$\mathcal{M}_{mn} = (-1)^{m-n} \frac{m}{n} \mathcal{M}_{nm}, \qquad (3.17)$$

where use has been made of

$$e^{iz\cos\varphi} = \sum_{n=-\infty}^{\infty} i^n \mathcal{J}_n(z) e^{in\varphi},$$

$$\mathcal{J}_{n-1}(z) + \mathcal{J}_{n+1}(z) = \frac{2n}{z} \mathcal{J}_n(z). \qquad (3.18)$$

Here $\mathcal{J}_n(z)$ is the Bessel function of the first kind of order n.

4 COHERENT BEAM-BEAM RESONANCES

Equation (3.12) can be formally solved to obtain the one-turn transfer map

$$\widetilde{\mathcal{R}}_k(n; 2\pi) = \exp(-2\pi i n \widetilde{\nu}_k) \times \left[\widetilde{\mathcal{R}}_k(n; 0) + \frac{\widetilde{\lambda}_k}{2} \sum_{m=-\infty}^{\infty} \mathcal{M}_{nm} \widetilde{\mathcal{R}}_{3-k}(m; 0) \right]. \qquad (4.1)$$

Consider now a coherent beam-beam resonance of the form

$$n_1 \widetilde{\nu}_1 + n_2 \widetilde{\nu}_2 = s + \Delta, \qquad (4.2)$$

where n_1, n_2 and s are integers, and Δ is the resonance detuning. Retaining only the $\pm n_1$ and the $\pm n_2$ elements in $\mathcal{M}_{nm}$, the transformation matrix of the coupled map equations (4.1) can be expressed as

$$\begin{pmatrix} e^{-i\psi_1} & 0 & \alpha_1 e^{-i\psi_1} & \alpha_1 e^{-i\psi_1} \\ 0 & e^{i\psi_1} & -\alpha_1 e^{i\psi_1} & -\alpha_1 e^{i\psi_1} \\ \alpha_2 e^{-i\psi_2} & \alpha_2 e^{-i\psi_2} & e^{-i\psi_2} & 0 \\ -\alpha_2 e^{i\psi_2} & -\alpha_2 e^{i\psi_2} & 0 & e^{i\psi_2} \end{pmatrix}, \qquad (4.3)$$

where

$$\psi_k = 2\pi n_k \widetilde{\nu}_k, \qquad \alpha_1 = \frac{\widetilde{\lambda}_1}{2} \mathcal{M}_{n_1 n_2}, \qquad (4.4)$$

$$\alpha_2 = \frac{\widetilde{\lambda}_2}{2} (-1)^{n_2 - n_1} \frac{n_2}{n_1} \mathcal{M}_{n_1 n_2}. \qquad (4.5)$$

The eigenvalues of the transfer matrix defined in Eq. (4.3) are the roots of the secular equation

$$\left(\lambda^2 - 2\lambda\cos\psi_1 + 1\right)\left(\lambda^2 - 2\lambda\cos\psi_2 + 1\right) + +2\alpha_1\alpha_2[\cos(\psi_1 - \psi_2) - \cos(\psi_1 + \psi_2)]\lambda^2 = 0. \qquad (4.6)$$

Casting Eq. (4.6) in the form

$$\left(\lambda^2 - 2c_1\lambda + 1\right)\left(\lambda^2 - 2c_2\lambda + 1\right) = 0, \qquad (4.7)$$

where

$$c_{1,2} = \frac{\cos\psi_1 + \cos\psi_2}{2} \pm \pm \frac{1}{2}\sqrt{(\cos\psi_1 - \cos\psi_2)^2 - 4A\sin\psi_1\sin\psi_2}, \qquad (4.8)$$

$$A = \frac{\widetilde{\lambda}_1 \widetilde{\lambda}_2}{4} (-1)^{n_2 - n_1} \frac{n_2}{n_1} \mathcal{M}_{n_1 n_2}^2, \qquad (4.9)$$

we obtain the stability criterion

$$|\cos\psi_1\cos\psi_2 + A\sin\psi_1\sin\psi_2| < 1. \qquad (4.10)$$

To conclude this section we note that in the case of a symmetric collider the stopbands calculated from Eq. (4.10) coincide with the results obtained by Chao and Ruth [see Eq. (31) of Ref. 1].

5 CONCLUDING REMARKS

Based on the Radon transform we have developed a macroscopic fluid model of the coherent beam-beam interaction. The linearized hydrodynamic equations are further solved and a stability criterion for coherent beam-beam resonances have been found in closed form.

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7 REFERENCES

[1] A.W. Chao and R.D. Ruth, *Particle Accelerators*, **16** 201 (1985).

[2] S.R. Deans, "*The Radon Transform and Some of Its Applications*", Wiley, New York 1992.

[3] Stephan I. Tzenov, *FERMILAB-Pub-98/275*, Batavia 1998.