TRANSVERSE ELECTRON-PROTON TWO-STREAM INSTABILITY IN A BUNCHED BEAM*

Tai-Sen F. Wang†, Paul J. Channell, Robert J. Macek, LANL, Los Alamos, NM 87545, USA
Ronald C. Davidson, PPPL, Princeton, NJ 08543, USA

Abstract

This paper is an analytical investigation of the transverse electron-proton (e-p) two-stream instability in a proton bunch propagating through a stationary electron background. The equations of motion, including the effect of damping, are derived for the centroids of the proton beam and the electron cloud. An approach is developed to solve the coupled linear centroid equations in the time domain describing the e-p instability in proton bunches with non-uniform line densities. Examples are presented for proton line densities corresponding to uniform and parabolic profiles.

1 INTRODUCTION

For intense proton beams, the focus of recent two-stream instability analyses has been on the transverse instability observed in the Proton Storage Ring (PSR) at Los Alamos National Laboratory [1-4]. The PSR stores a long proton bunch with a near triangular line density profile for a duration of about one millisecond. The instability is observed as rapidly growing transverse oscillations of the stored beam, usually occuring when the beam intensity reaches 2.5×10^{13} ppp or higher, causing fast beam loss. Experimental results support the conjecture that the instability in PSR is due to the two-stream interaction between the circulating proton beam and the electrons created in the ring, i.e., the so called e-p instability. However, the understanding of the physics of this instability is usually based on the theory developed for a continuous beam of uniform line density [5-10]. Although computer simulations have been implemented or are being developed [10,11] to study the e-p instability in bunched beams, a companion analytical theory still remains to be developed. The present work is an attempt to investigate the transverse e-p instability in a proton bunch using an analytical approach based on the centroid model built on the "one-pass" interaction between the protons and the electrons.

2 THEORETICAL MODEL

We consider a bunched proton beam of full length L and circular cross section of radius a, propagating with a constant velocity **v** through a stationary electron background of infinite extent in the direction of beam propagation. We assume that in the equilibrium state, the electrons are trapped in the proton beam, and the particles are distributed uniformly in the transverse direction so the trapped electrons experience a linear transverse focusing force. The protons are confined in the transverse direction by a linear external focusing force. A Cartesian coordinate system is chosen such that the z axis is pointing opposite to the direction of proton propagation, and the origin coincides with the center of the beam cross section. The line densities of the protons and electrons, λ_p and λ_e, generaly depend on z. The synchrotron motion of the protons and the axial motion of the electrons in the laboratory frame are neglected for simplicity. Further simplifications are made by neglecting the impedance due to the beam environment, like the beam pipe, and by considering the transverse motion in only one direction, say the y direction. The stability study is based on a model in which each electron interacts with the proton beam only once, i.e., a "one-pass" interaction between the electrons and protons.

The centroid of the proton beam $Y_p(z,t)$ and the centroid of electrons $Y_e(z,t)$ are defined by

$$Y_q(z,t) = \int_{-\infty}^{\infty} y_q(z,t,\omega_q) F_q(\omega_q) d(\omega_q/\Delta_q) \ , \qquad (1)$$

where the subscripts q stands for p (protons) or e (electrons), $y_q(z,t,\omega_q)$ is the particle displacement at the position z and time t, ω_q is the oscillation frequency, $F_q(\omega_q)$ is the frequency distribution function, and Δ_q characterizes the frequency spread of ω_q. The equation of motion for a single particle is

$$\ddot{y}_q = -\omega_q^2 y_q + (\hat{F}_q/m_q) \ , \qquad (2)$$

where the super-dot denotes derivative with respect to time, $\hat{F}_q$ is the perturbing force, and m_q is the relativistic mass of a proton or an electron. For the perturbing forces, we make the approximations $\hat{F}_e/m_e = \Omega^2(z) Y_p$ and $\hat{F}_p/m_p = G(z) Y_e$, where $\Omega(z) = (c/a)\sqrt{2 r_e \lambda_p(z)}$ is the electron bounce frequency, $G(z) = \omega_\beta^2 \xi(z)$, $\xi(z) = 2 r_p c^2 \lambda_e(z)/(a^2 \omega_\beta^2 \gamma)$, c is the speed of light, ω_β is the betatron frequency, $\gamma = (1 - v^2/c^2)^{-1/2}$, and r_q is the classical radius of a proton or an electron.

We consider a Lorentzian distribution function $F_q(\omega_q) = (\Delta_q^2/\pi)[\Delta_q^2 + (\omega_q - \omega_{qo})^2]^{-1}$, where ω_{qo} is the mean value of ω_q, and assume that the general solution to Eq. (2) can be expressed as a linear combination of $e^{i\omega_q t}$ and $e^{-i\omega_q t}$, where $i = \sqrt{-1}$. Averaging Eq. (2) yields

$$D^2 Y_p + 2\Delta_p D Y_p + (\omega_\beta^2 + \Delta_p^2) Y_p = G(z) Y_e \ , \qquad (3)$$

* Research supported by Los Alamos National Laboratory under the auspices of the US Department of Energy, and in part by the Spallation Neutron Source (SNS) Project.

† TWANG@LANL.GOV

and

$$\ddot{Y}_e + 2\Delta_e \dot{Y}_e + [\Omega^2(z) + \Delta_e^2] Y_e = \Omega^2(z) Y_p \ , \qquad (4)$$

where we have replaced the time derivative in Eq. (2) by $D = \partial/\partial t - v(\partial/\partial z)$ for the moving proton beam in the laboratory frame. In deriving Eqs. (3) and (4), we have also assumed that the incoherent betatron frequency shift due to the self-fields of the proton beam is negligible, and that the maximum value of λ_e is much smaller than that of λ_p, so that $\omega_{po} = \omega_\beta$ = undepressed betatron frequency. Note that the damping factors in Eqs. (3) and (4) depend on the choice of the frequency distribution function.

3 STABILITY ANALYSIS

Since the analysis here is based on the linearized equations (3) and (4), the subsequent analysis is necessarily limited to the linear regime. The axial coordinates in the laboratory and beam frames are denoted by z and z', respectively. We assume that the origins of these two coordinate systems coincide when $t = 0$, and that the head of the proton bunch is located at $z' = 0$. For simplicity, relativistic effects are neglected and the Galilean transformation $z' = z + vt$ is used between coordinate systems.

Substituting $Y_p(z', t) = \mathrm{e}^{-\Delta_p t} Y_{pn}(z', t)$ and $Y_e(z,t) = \mathrm{e}^{-\Delta_e t} Y_{en}(z,t)$ into Eqs. (3) and (4), and tranforming the proton equation to the beam frame, yield

$$\ddot{Y}_{pn} + \omega_\beta^2 Y_{pn} = G(z') \mathrm{e}^{(\Delta_p - \Delta_e)t} Y_{en} \ , \qquad (5)$$

and

$$\ddot{Y}_{en} + \Omega^2(t - t_e) \big[Y_{en} - \mathrm{e}^{-(\Delta_p - \Delta_e)t} Y_{pn} \big] = 0 \ , \qquad (6)$$

where a sharp-edged line density λ_p is assumed so that $\Omega(x) = 0$ for $x \le 0$, and $t_e = -z/v$ is the time when a slice of electrons located at the position z in the laboratory frame enters the proton bunch. Using the variational method and applying a Fourier transformation, we derive from Eqs. (5) and (6) the integral equation

$$\hat{Y}(z', k) = \int_0^{z'/v} \frac{\Omega^2(x) G(vx)}{(\omega_\beta^2 - k^2) W(x)} \big[\Phi(s') \Psi(x) - \Psi(s') \Phi(x) \big] \hat{Y}(vx, k) \, dx \ , \qquad (7)$$

where $\hat{Y}(z', k) = \mathrm{e}^{(ik - \Delta_p + \Delta_e)s'} \tilde{Y}(z', k)/G(z')$, $s' = z'/v \le L/v$, and $\tilde{Y}(z', k)$ is the Fourier transform of $Y_{pn}(z', t)$. Here, $\Phi(t) = \Phi(z, t)$ and $\Psi(t) = \Psi(z, t)$ are the linearly independent solutions of the homogeneous part of Eq. (6), and $W(x)$ is the Wronskian of $\Phi(x)$ and $\Psi(x)$. In obtaining Eq. (7), we have assumed that $Y_{en} = dY_{en}/dt = 0$ for $t \le t_e$. Differentiating Eq. (7) twice leads to the following equation for $\hat{Y}(z', k)$, i.e.,

$$v^2 d^2 \hat{Y}/dz'^2 = -\Omega^2(s') \big\{ 1 + [G(z')/(k^2 - \omega_\beta^2)] \big\} \hat{Y} \ . \qquad (8)$$

Thus, the stability analysis has now resulted in solving Eqs. (7) or (8) and inverting a Fourier transformation. For non-uniform line densities, Eqs. (7) or (8) have exact solutions only for very limited cases. A possible approximation in Eq. (7) can be seen by substituting $\hat{Y}(z', k) = \zeta(z', k) \Phi(s')$ into Eq. (7), which gives

$$\zeta(z', k) = \int_0^{z'/v} \frac{\zeta(vx, k) \Omega^2(x) G(vx) \Phi(x) \Psi(x)}{(\omega_\beta^2 - k^2) W(x)} \times \big\{ 1 - [\Psi(s')/\Phi(s')][\Phi(x)/\Psi(x)] \big\} \, dx \ . \qquad (9)$$

We assume that (i) Φ and Ψ are such that $\Phi\Psi$ is a smooth function and Φ/Ψ is an oscillatory function, and (ii) that the main motion described in $\zeta(z', k)$ are betatron oscillations at much lower frequency than the electron bounce motion in the beam, so that the contribution from the fast oscillatory term containing Φ/Ψ in the integration in Eq. (9) is negligibly small. Then, neglecting the term proportional to Φ/Ψ in Eq. (9) leads to the approximate solution

$$Y_p(z', t) \sim \big\{ J(z')/[\omega_\beta(t - s')]^3 \big\}^{1/4} G(z') \Phi(s') \times \exp\Big[-(i\omega_\beta + \Delta_p)(t - s') - \Delta_e s' - (i/4) J(z') + \sqrt{2\omega_\beta J(z')(t - s')} \ \Big] \ , \qquad (10)$$

where

$$J(z') = i \int_0^{z'/v} \frac{\Omega^2(x) G(vx)}{\omega_\beta^2 W(x)} \Phi(x) \Psi(x) dx \ . \qquad (11)$$

Similarly, the solution for $Y_p(z', t)$ in terms of $\Psi(s')$ can be derived by making the substitution $\hat{Y}(z', k) = \zeta(z', k) \Psi(s')$ in Eq. (7). Equation (10) indicates that the instability grows in both time and space consistent with computer simulation results[11]. In the absence of damping, the asymptotic temporal growth of the instability in the beam frame is proportional to $\mathrm{e}^{\alpha t^{1/2}}$ (where the constant α is independent of t), and the spatial growth is determined primarily by the quantity $J(z')$. Further, we see from Eq. (10) that the e-p mode "wiggles" in space proportional to $\Phi \mathrm{e}^{-iJ/4}$. Note that Eq. (10) also indicates that the perturbation is eventually damped as long as Δ_p is nonzero. This is due to the combined effects of finite proton bunch length and the "one-pass" electron-proton interaction.

The following two examples illustrate the applications of the theory developed here.

Example A: Uniform Line Densities λ_p and λ_e

When both λ_p and λ_e are uniform, $\Omega^2(x) = \omega^2$, and $G(x) = \xi \omega_\beta^2$, where ω is the electron bounce frequency, and ξ is a constant. Equations (7) and (8) have the exact solutions $\Phi(z') = \exp(i\omega s' \eta)$ and $\Psi(z') = \exp(-i\omega s' \eta)$, where $\eta = \{1 + [\xi \omega_\beta^2/(k^2 - \omega_\beta^2)]\}^{1/2}$. We concentrate on the part of the solution containing Φ here. The part containing Ψ can be treated in the same manner. The solution for $Y_p(z',t)$ is

$$Y_p(z',t) \sim \xi \omega_\beta^2 \mathrm{e}^{\Delta_p(s' - t) - \Delta_e s'} \int_{-\infty}^{\infty} \mathrm{e}^{ik(t - s') + i\omega s' \eta} dk \ , \qquad (12)$$

where $z' \le L$. The inverse Fourier transformation in Eq. (12) can be carried out analytically only in a few parameter ranges by using the steepest descent method. We limit discussion here to two cases.

The first case corresponds to $\xi^2 \ll \xi \omega s'/[\omega_\beta(t - s')] \ll 1$. In this range, the unstable mode has the approximate solution

$$Y_p(z',t) \sim \left\{\omega s'/[\omega_\beta(t-s')]^3\right\}^{1/4}\exp\Big[-\Delta_e s' - i\omega_\beta t + i(\omega+\omega_\beta)s' - \Delta_p(t-s') + \sqrt{\xi\omega\omega_\beta s'(t-s')}\,\Big] , \quad (13)$$

where we have neglected constant factors in the amplitude. This solution agrees well with the result of applying the approximation in Eq. (10). Equation (13) indicates that the proton beam "wiggles" in space approximately according to the electron bounce frequency as seen in experiments[1-4]. It can be shown that if the damping is large enough, the instability grows to a maximum amplitude and then subsequently damps.

The second case corresponds to $\xi\omega^2 {s'}^2/[\omega_\beta(t-s')]^2 \ll \xi^3 \ll 1$. The approximate solution for the unstable mode in this range is found to be

$$Y_p(z',t) \sim \left\{(\omega s')^2/[\omega_\beta(t-s')]^5\right\}^{1/6}\exp\Big\{(i\omega-\Delta_e)s' + (3/4)(\sqrt{3}-i)\big[\xi\omega^2\omega_\beta {s'}^2(t-s')\,\big]^{1/3} -(i\omega_\beta+\Delta_p)(t-s')\,\Big\} . \quad (14)$$

This solution shows that when there is no damping, the instability grows like $\mathrm{e}^{\alpha t^{1/3}}$ - a result similar to that obtained in earlier works [12-14]. In the presence of damping, an instability continuing to grow from the previous parameter range will be damped in this regime after the perturbation reaches a maximum amplitude. In reality, the growth of the instability can stop due to substantial loss of beam particles before it reaches the theoretical maximum, or the growth can saturate at a lower level due to nonlinear effects not discussed here.

Example B: Parabolic λ_p and Uniform λ_e

In this case, $G(x) = \xi\omega_\beta^2$ is a constant, and $\Omega^2(z'/v) = 4\omega_o^2\big[(v/L)(z'/v) - (v/L)^2(z'/v)^2\big]$, where ω_o is the electron bounce frequency at the center of the proton bunch. The homogeneous part of Eq. (6) becomes

$$d^2Y_{en}/dp^2 + \big[h - (p^2/4)\big]Y_{en} = 0 , \quad (15)$$

where $h = L\omega_o/(4v)$, and $p = 2\sqrt{\omega_o v/L}\big[s' - L/(2v)\big]$. The two linearly independent solutions to Eq. (15), $\Phi(s')$ and $\Psi(s')$, are chosen to be

$$\Phi = \Psi^* = U(-h,p) + i\Gamma(2\mu)V(-h,p) = R\mathrm{e}^{i\Theta} , \quad (16)$$

where $*$ denotes the complex conjugate, $U(x,y)$ and $V(x,y)$ are parabolic cylinder functions[15], $\Gamma(x)$ is the Gamma function, and $\mu = (1/4) - (h/2)$. Here, R and Θ have the following expansions valid for $|p| \ll 2\sqrt{h}$ [15],

$$R \approx \frac{\Gamma(\mu+h)}{2^\mu\sqrt{\pi}}\Big(1 + \frac{p^2}{16h} + \frac{5p^4}{512h^2} + \cdots\Big) , \quad (17)$$

and

$$\Theta \approx \pi\mu + \sqrt{h}\,p\big\{1 - [p^2/(24h)] + \cdots\big\} . \quad (18)$$

Substituting Eqs. (16) - (18) into Eq. (11), using the Wronskian relation between $U(x,y)$ and $V(x,y)$, and retaining the three lowest-order terms in z'/L, we obtain

$$J(z') \approx 3\xi h^{3/2}(z'/L)^2\big\{1 - [10z'/(9L)] + (z'/L)^2\big\} , \quad (19)$$

where only two lowest-order terms in the expansion of R^2 have been retained. The solution for $Y_p(z',t)$ can be obtained by substituting Eqs. (16) and (19) into Eq. (10). For no damping, the combined spatial and temporal growths at the center and the tail of the bunch are approximately $\exp[(\xi\omega_\beta t)^{1/2}h^{3/4}]$ and $\exp[2.3(\xi\omega_\beta t)^{1/2}h^{3/4}]$, respectively.

4 CONCLUSIONS

We have derived the equations for the transverse motion of the centroids of the proton bunch and the electrons. The damping effect was included by considering Lorentzian distributions of the particles' transverse oscillation frequencies. Based on the model of "one-pass" interaction between the protons and the electrons, we discussed approaches for solving the linearized centroid equations to investigate the e-p instability in finite-length proton bunches with arbitrary line density. Case studies were presented for uniform and parabolic proton line densities. It was found that the e-p instability can grow in both space and time. Depending on the stage of the instability, the temporal growth of the electron-proton oscillations can be proportional to $\mathrm{e}^{\alpha t^{1/2}}$ or $\mathrm{e}^{\alpha t^{1/3}}$. For a Lorentzian distribution of proton oscillation frequencies with nonzero spread, the instability will eventually be damped for long times.

5 REFERENCES

[1] R. Macek, et al., these proceedings, and presentation at ICANS-XV, Nov. 6-9, 2000, KEK, Tsukuba, Japan, to be published in the Proceedings.

[2] Presentations at the ICFA 8th Beam Dynamics Mini-Workshop, Feb. 16 - 18, 2000, Santa Fe, NM, Session I. http://www.aps.anl.gov/conferences/icfa/two-stream.html

[3] Proc. of the Workshop on Electron Effects, ed. T. F. Wang and A. Jason, LANL Report LA-UR-98-1601, May 1998.

[4] D. Neuffer, et al., Nucl. Instrum. Meth., **A 321** (1992) 1.

[5] R. C. Davidson, et al., Phys. Rev. STAB, **2** (1999) 054401.

[6] L. J. Laslett, A. M. Sessler and D. Möhl, Nucl. Instrum. Meth., **121** (1974) 517.

[7] H. Schönauer and B. Zotter, CERN Internal Note, 1972.

[8] D. G. Koshkarev and P. R. Zenkevich, Particle Accelerators, **3** (1972) 1.

[9] E. Keil and B. Zotter, CERN-ISR-TH/71-58, Dec. 1971.

[10] H. Qin, R. C. Davidson, Edward Startsev, and W. W. Lee, these proceedings, also see H. Qin et al., Phys. Rev. STAB, **3** (2000) 084401; **3** (2000) 109901.

[11] T. Wang, AIP Conf. Proc. 496, p. 305 (1999).

[12] M. Lampe et al., Phys. Fluids, **B5** (1993) 1888.

[13] D. H. Whittum, et al., Phys. Rev. Lett., **67** (1991) 991.

[14] R. J. Briggs, in Quarterly Progress Report No. 85 of Research Laboratory of Electronics, MIT, p. 183, 1967.

[15] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, US NBS, 1964, Chap. 19.