New Journal of Physics
An Institute of Physics and Deutsche Physikalische Gesellschaft Journal

Renormalization group reduction of non-integrable Hamiltonian systems

Stephan I Tzenov
Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543, USA
E-mail: stzenov@pppl.gov

New Journal of Physics **4** (2002) 6.1–6.12 (http://www.njp.org/)
Received 8 August 2001, in final form 27 November 2001
Published 19 February 2002

Abstract. Based on the renormalization group (RG) method, a reduction of non-integrable multi-dimensional Hamiltonian systems has been performed. The evolution equations for the slowly varying part of the angle-averaged phase space density and for the amplitudes of the angular modes have been derived. It has been shown that these equations are precisely the RG equations. As an application of the approach developed, the modulational diffusion in a one-and-a-half-degree-of-freedom dynamical system has been studied in detail.

1. Introduction

It is well known that dynamical systems may exhibit irregular motion in certain regions of phase space [1, 2]. These regions differ in size, from being considerably small, to occupying large parts of phase space. This depends mostly on the strength of the perturbation, as well as on the intrinsic characteristics of the system. For comparatively small perturbations the regularity of the motion is expressed in the existence of adiabatic action invariants. In the course of nonlinear interaction the action invariants vary within a certain range, prescribed by the integrals of motion (if such exist). For chaotic systems some (or all) of the integrals of motion are destroyed, causing specific trajectories to become extremely complicated. These trajectories look random in their behaviour, therefore it is natural to explore the statistical properties of chaotic dynamical systems.

Much experimental and theoretical evidence [3, 4] of nonlinear effects characterizing the dynamics of particles in accelerators and storage rings is available at present. An individual particle propagating in an accelerator experiences growth of amplitude of betatron oscillations in a plane transverse to the particle orbit, whenever a perturbing force acts on it. This force may be of various origins, for instance high-order multipole magnetic field errors, space charge forces, beam–beam interaction force, power supply ripple or other external and collective forces. Therefore, the Hamiltonian governing the motion of a beam particle is far from being integrable,

1367-2630/02/000006+12$30.00

and an irregular behaviour of the beam is clearly observed, especially for a large number of revolutions.

A rich arsenal of analytical methods to study effects of nonlinear behaviour of beams in accelerators and storage rings, ranging from classical perturbation theory [2, 5] to the Lie algebraic approach [6]–[8], is available at present. The recently developed renormalization group (RG) method has been successfully applied to both continuous dynamical systems [9]–[11] and maps [12, 13], that are of general interest in the physics of accelerators and beams. The advantage of the RG method is embedded in the fact that it is equally powerful to study finite-dimensional, as well as continuous systems. This makes it particularly useful when applied to analyse the properties of chaotic dynamical systems in both the stability region and the globally stochastic region in phase space.

The idea to treat the evolution of chaotic dynamical systems in a statistical sense is not new; many rigorous results related to the statistical properties of such systems can be found in [14]. Many of the details concerning the transport phenomena in the space of adiabatic action invariants only are also well understood [2]. In this aspect the results presented here are in a sense re-derivation of previously obtained ones by means of a different method. What is new, however, is the approach followed to obtain the diffusion properties in action variable space, as well as a new evolution equation for the angle-dependent part of the phase space density. Furthermore, instead of the widely used phenomenological method to derive the diffusion coefficient (tensor), the procedure pursued in the present paper is a more consistent one, with a starting point the Liouville equation for the phase space density.

We first employ the projection operator method of Zwanzig [15] to derive the equations for the two parts of the phase space density: the part averaged over the angle variables F, and the remainder G (see equation (2.10) in the next section). As expected, the two equations are coupled. Next we extract the relevant long-timescale behaviour embedded in the equations for F and G by means of the RG method [9]–[11]. It is remarkable, and at the same time not surprising that the equations governing the long-timescale dynamics are the renormalization group equations (RGEs). These are obtained in section 4 by means of renormalizing the perturbative solution to the equations for F and G (see equations (2.22) and (2.23) of section 2). Finally, in section 5 a one-dimensional example of a chaotic system exhibiting weak instability (the so-called modulational diffusion) is considered to demonstrate the approach developed here.

2. Projection operator method

Single-particle dynamics in cyclic accelerators and storage rings is most properly described by the adiabatic action invariants (Courant–Snyder invariants [16, 17]) and the angle variables canonically conjugate to them. However, to be more general we consider here a dynamical system with N degrees of freedom, governed by the Hamiltonian written in action-angle variables $(\boldsymbol{J}, \boldsymbol{\alpha})$ as

$$H(\boldsymbol{\alpha}, \boldsymbol{J}; \theta) = H_0(\boldsymbol{J}) + \epsilon V(\boldsymbol{\alpha}, \boldsymbol{J}; \theta), \tag{2.1}$$

where θ is the independent azimuthal variable (widely used in accelerator physics), playing the role of time and $\boldsymbol{J}$ and $\boldsymbol{\alpha}$ are N-dimensional vectors

$$\boldsymbol{J} = (J_1, J_2, \ldots, J_N), \qquad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N). \tag{2.2}$$

Moreover $H_0(\boldsymbol{J})$ is the integrable part of the Hamiltonian, ϵ is a formal small parameter, while $V(\boldsymbol{\alpha}, \boldsymbol{J}; \theta)$ is the perturbation periodic in the angle variables

$$V(\boldsymbol{\alpha}, \boldsymbol{J}; \theta) = \sum_{\boldsymbol{m}}{}' V_{\boldsymbol{m}}(\boldsymbol{J}; \theta) \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{\alpha}), \tag{2.3}$$

where $\sum'$ denotes exclusion of the harmonic $\boldsymbol{m} = (0, 0, \ldots, 0)$ from the above sum. The Hamilton equations of motion are

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}\theta} = \omega_{0k}(\boldsymbol{J}) + \epsilon \frac{\partial V}{\partial J_k}, \qquad \frac{\mathrm{d}J_k}{\mathrm{d}\theta} = -\epsilon \frac{\partial V}{\partial \alpha_k}, \tag{2.4}$$

where

$$\omega_{0k}(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_k}. \tag{2.5}$$

In what follows (in particular in section 4) we assume that the nonlinearity coefficients

$$\gamma_{kl}(\boldsymbol{J}) = \frac{\partial^2 H_0}{\partial J_k \partial J_l} \tag{2.6}$$

are small and can be neglected. The Liouville equation governing the evolution of the phase space density $P(\boldsymbol{\alpha}, \boldsymbol{J}; \theta)$ can be written as

$$\frac{\partial}{\partial \theta} P(\boldsymbol{\alpha}, \boldsymbol{J}; \theta) = [\widehat{\mathcal{L}}_0 + \epsilon \widehat{\mathcal{L}}_v(\theta)] P(\boldsymbol{\alpha}, \boldsymbol{J}; \theta). \tag{2.7}$$

Here the operators $\widehat{\mathcal{L}}_0$ and $\widehat{\mathcal{L}}_v$ are given by the expressions

$$\widehat{\mathcal{L}}_0 = -\omega_{0k}(\boldsymbol{J}) \frac{\partial}{\partial \alpha_k}, \qquad \widehat{\mathcal{L}}_v = \frac{\partial V}{\partial \alpha_k} \frac{\partial}{\partial J_k} - \frac{\partial V}{\partial J_k} \frac{\partial}{\partial \alpha_k}, \tag{2.8}$$

where summation over repeated indices is implied. Next we define the projection operator onto the subspace of action variables by the following integral:

$$\widehat{\mathcal{P}} f(\boldsymbol{J}; \theta) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \mathrm{d}\alpha_1 \cdots \int_0^{2\pi} \mathrm{d}\alpha_N f(\boldsymbol{\alpha}, \boldsymbol{J}; \theta), \tag{2.9}$$

where $f(\boldsymbol{\alpha}, \boldsymbol{J}; \theta)$ is a generic function of its arguments. Let us introduce also the functions

$$F = \widehat{\mathcal{P}} P, \qquad G = (1 - \widehat{\mathcal{P}}) P = \widehat{\mathcal{C}} P, \qquad (P = F + G). \tag{2.10}$$

From equation (2.7) with the obvious relations

$$\widehat{\mathcal{P}} \widehat{\mathcal{L}}_0 = \widehat{\mathcal{L}}_0 \widehat{\mathcal{P}} \equiv 0, \qquad \widehat{\mathcal{P}} \widehat{\mathcal{L}}_v \widehat{\mathcal{P}} \equiv 0 \tag{2.11}$$

in hand it is straightforward to obtain the equations

$$\frac{\partial F}{\partial \theta} = \epsilon \widehat{\mathcal{P}} \widehat{\mathcal{L}}_v G = \epsilon \frac{\partial}{\partial J_k} \widehat{\mathcal{P}} \left(\frac{\partial V}{\partial \alpha_k} G \right), \tag{2.12}$$

$$\frac{\partial G}{\partial \theta} = \widehat{\mathcal{L}}_0 G + \epsilon \widehat{\mathcal{C}} \widehat{\mathcal{L}}_v G + \epsilon \widehat{\mathcal{L}}_v F. \tag{2.13}$$

Our goal in the subsequent exposition is to analyse equations (2.12) and (2.13) using the RG method. It will prove efficient to eliminate the dependence on the angle variables in G and V by noting that the eigenfunctions of the operator $\widehat{\mathcal{L}}_0$ form a complete set, so that every function periodic in the angle variables can be expanded in this basis. Using Dirac's 'bra–ket' notation we write

$$|\boldsymbol{n}\rangle = \frac{1}{(2\pi)^{N/2}} \exp(\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\alpha}), \qquad \langle\boldsymbol{n}| = \frac{1}{(2\pi)^{N/2}} \exp(-\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\alpha}). \tag{2.14}$$

The projection operator $\widehat{\mathcal{P}}$ can be represented in the form [18]

$$\widehat{\mathcal{P}} = \widehat{\mathcal{P}}_{\boldsymbol{0}} = |\boldsymbol{0}\rangle\langle\boldsymbol{0}|. \tag{2.15}$$

One can also define a set of projection operators $\widehat{\mathcal{P}}_{\boldsymbol{n}}$ according to the expression [18]

$$\widehat{\mathcal{P}}_{\boldsymbol{n}} = |\boldsymbol{n}\rangle\langle\boldsymbol{n}|. \tag{2.16}$$

It is easy to check the completeness relation

$$\sum_{\boldsymbol{n}} \widehat{\mathcal{P}}_{\boldsymbol{n}} = 1, \tag{2.17}$$

from which and from equation (2.15) it follows that

$$\widehat{\mathcal{C}} = \sum_{\boldsymbol{n}\neq\boldsymbol{0}}{}' |\boldsymbol{n}\rangle\langle\boldsymbol{n}|. \tag{2.18}$$

Decomposing the quantities F, G and V in the basis (2.14) as

$$F = F(\boldsymbol{J};\theta)|\boldsymbol{0}\rangle, \tag{2.19}$$

$$G(\boldsymbol{\alpha},\boldsymbol{J};\theta) = \sum_{\boldsymbol{m}\neq\boldsymbol{0}}{}' G_{\boldsymbol{m}}(\boldsymbol{J};\theta)|\boldsymbol{m}\rangle, \tag{2.20}$$

$$V(\boldsymbol{\alpha},\boldsymbol{J};\theta) = \sum_{\boldsymbol{n}\neq\boldsymbol{0}}{}' V_{\boldsymbol{n}}(\boldsymbol{J};\theta)|\boldsymbol{n}\rangle, \tag{2.21}$$

from equations (2.12) and (2.13) we obtain

$$\frac{\partial F}{\partial\theta} = \mathrm{i}\epsilon\frac{\partial}{\partial J_k}\left(\sum_{\boldsymbol{n}}{}' n_k V_{\boldsymbol{n}} G_{-\boldsymbol{n}}\right), \tag{2.22}$$

$$\frac{\partial G_{\boldsymbol{n}}}{\partial\theta} = -\mathrm{i}n_k\omega_{0k}G_{\boldsymbol{n}} + \mathrm{i}\epsilon\sum_{\boldsymbol{m}}{}' \left[n_k V_{\boldsymbol{n}-\boldsymbol{m}}\frac{\partial G_{\boldsymbol{m}}}{\partial J_k} - m_k\frac{\partial}{\partial J_k}(V_{\boldsymbol{n}-\boldsymbol{m}}G_{\boldsymbol{m}})\right] + \mathrm{i}\epsilon n_k V_{\boldsymbol{n}}\frac{\partial F}{\partial J_k}. \tag{2.23}$$

A similar harmonic expansion of the Liouville equation has been used by Zaslavsky [19], who derived a master kinetic equation for the one-dimensional nonlinear oscillator. Equations (2.22) and (2.23) comprise the starting point for the RG analysis outlined in section 4. We are primarily interested in the long-time evolution of the original system governed by certain amplitude equations. These will turn out to be precisely the RGEs.

3. Renormalization group reduction of Hamilton's equations

Let us consider the solution to Hamilton's equations of motion (2.4) for a small perturbation parameter ϵ. It is natural to introduce the naive perturbation expansion

$$\alpha_k = \alpha_k^{(0)} + \epsilon\alpha_k^{(1)} + \epsilon^2\alpha_k^{(2)} + \cdots, \qquad J_k = J_k^{(0)} + \epsilon J_k^{(1)} + \epsilon^2 J_k^{(2)} + \cdots. \tag{3.1}$$

The lowest-order perturbation equations have the trivial solution

$$\alpha_k^{(0)} = \omega_{0k}\theta + \varphi_k, \qquad J_k^{(0)} = A_k, \tag{3.2}$$

where A_k and φ_k are constant amplitude and phase, respectively. We write the first-order perturbation equations as

$$\frac{\mathrm{d}\alpha_k^{(1)}}{\mathrm{d}\theta} = \gamma_{kl}(\boldsymbol{A})J_l^{(1)} + \frac{\partial V}{\partial A_k}, \qquad \frac{\mathrm{d}J_k^{(1)}}{\mathrm{d}\theta} = -\frac{\partial V}{\partial \alpha_k^{(0)}}. \tag{3.3}$$

Assuming that the modes $V_{\boldsymbol{n}}(\boldsymbol{J};\theta)$ are periodic in θ, we can expand them in a Fourier series

$$V_{\boldsymbol{n}}(\boldsymbol{J};\theta) = \sum_{\mu=-\infty}^{\infty} V_{\boldsymbol{n}}(\boldsymbol{J};\mu)\exp(\mathrm{i}\mu\nu_{\boldsymbol{n}}\theta). \tag{3.4}$$

If the original system (2.1) is far from primary resonances of the form

$$n_k^{(\mathrm{R})}\omega_{0k} + \mu\nu_R = 0 \tag{3.5}$$

we can solve the first-order perturbation equations (3.3), yielding the result

$$\begin{aligned} \alpha_k^{(1)} = {} & \mathrm{i}\sum_{\boldsymbol{m}}{}'\sum_{\mu}\gamma_{kl}(\boldsymbol{A})m_l V_{\boldsymbol{m}}(\mu)\frac{\exp[\mathrm{i}(m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}})\theta]}{(m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}})^2}\exp(\mathrm{i}m_s\varphi_s) \\ & -\mathrm{i}\sum_{\boldsymbol{m}}{}'\sum_{\mu}\frac{\partial V_{\boldsymbol{m}}(\mu)}{\partial A_k}\frac{\exp[\mathrm{i}(m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}})\theta]}{m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}}}\exp(\mathrm{i}m_s\varphi_s), \end{aligned} \tag{3.6}$$

$$J_k^{(1)} = -\sum_{\boldsymbol{m}}{}'\sum_{\mu} m_k V_{\boldsymbol{m}}(\mu)\frac{\exp[\mathrm{i}(m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}})\theta]}{m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}}}\exp(\mathrm{i}m_s\varphi_s). \tag{3.7}$$

The second-order perturbation equations have the form

$$\frac{\mathrm{d}\alpha_k^{(2)}}{\mathrm{d}\theta} = \gamma_{kl}(\boldsymbol{A})J_l^{(2)} + \frac{1}{2}\frac{\partial\gamma_{kl}}{\partial A_s}J_l^{(1)}J_s^{(1)} + \frac{\partial^2 V}{\partial A_k\partial A_l}J_l^{(1)} + \frac{\partial^2 V}{\partial A_k\partial\alpha_l^{(0)}}\alpha_l^{(1)}, \tag{3.8}$$

$$\frac{\mathrm{d}J_k^{(2)}}{\mathrm{d}\theta} = -\frac{\partial^2 V}{\partial\alpha_k^{(0)}\partial A_l}J_l^{(1)} - \frac{\partial^2 V}{\partial\alpha_k^{(0)}\partial\alpha_l^{(0)}}\alpha_l^{(1)}. \tag{3.9}$$

The solution to equation (3.9) reads as

$$\begin{aligned} J_k^{(2)} = {} & 2\pi\mathcal{R}(\theta)\sum_{\boldsymbol{m}>\boldsymbol{0}}{}'\sum_{\mu} m_k m_l\frac{\partial|V_{\boldsymbol{m}}(\mu)|^2}{\partial A_l}\Re(\gamma; m_s\omega_{0s}+\mu\nu_{\boldsymbol{m}}) \\ & +2\pi\mathcal{R}(\theta)\sum_{\boldsymbol{m}>\boldsymbol{0}}{}'\sum_{\mu} m_k m_l m_s\gamma_{ls}(\boldsymbol{A})|V_{\boldsymbol{m}}(\mu)|^2\left.\frac{\partial}{\partial a}\Re(\gamma;a)\right|_{a=m_r\omega_{0r}+\mu\nu_{\boldsymbol{m}}} \\ & +\text{oscillating terms}, \end{aligned} \tag{3.10}$$

where

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\theta} = 1, \tag{3.11}$$

$$\pi\Re(x;y) = \frac{x}{x^2+y^2}, \qquad \lim_{x\to 0}\Re(x;y) = \delta(y). \tag{3.12}$$

Here $-\mathrm{i}\gamma$ is a small imaginary quantity, which has been added *ad hoc* in the denominators of the expressions (3.6) and (3.7) for the purpose of regularization. The limit $\gamma \to 0$ will be taken in the final result. Note that the formal procedure used here is equivalent to utilizing the well known Plemelj formula.

As expected, in the second-order perturbation solution (3.10) the first and the second terms are secular, because $\mathcal{R}(\theta) = \theta$. To remove these secularities we follow the general prescription of the RG method [9, 10]. Let us also note that an alternative formulation of the RG method in terms of invariant manifolds and classical theory of envelopes [11] exists. First, we select the slowly varying part of the perturbation solution governing the long-time evolution of the system. Up to second order in the perturbation parameter ϵ it consists of the constant zero-order term A_k and the second-order secular terms. Next, we introduce the intermediate time τ, and in order to absorb the difference $\tau = \theta - (\theta - \tau)$ we make the new renormalized amplitude $A_k(\tau)$ dependent on τ. Since the long-time solution thus constructed should not depend on τ its derivative with respect to τ must be equal to zero. This also holds for $\tau = \theta$, so that finally

$$\frac{\mathrm{d}A_k(\theta)}{\mathrm{d}\theta} = 2\pi\epsilon^2 \sum_{\boldsymbol{m}>\boldsymbol{0}}{}' \sum_{\mu} m_k m_l \frac{\partial |V_{\boldsymbol{m}}(\boldsymbol{A};\mu)|^2}{\partial A_l} \Re(\gamma; m_s\omega_{0s} + \mu\nu_{\boldsymbol{m}})$$
$$+2\pi\epsilon^2 \sum_{\boldsymbol{m}>\boldsymbol{0}}{}' \sum_{\mu} m_k m_l m_s \gamma_{ls}(\boldsymbol{A}) |V_{\boldsymbol{m}}(\boldsymbol{A};\mu)|^2 \frac{\partial}{\partial a} \Re(\gamma; a)\bigg|_{a=m_r\omega_{0r}+\mu\nu_{\boldsymbol{m}}}. \quad (3.13)$$

Equation (3.13) is known as the RGE. It describes the slow long-time evolution of the action variables.

One may proceed further with solving equation (3.8). Its solution will contain secular terms as well, which can be removed exactly in the same manner as done for equation (3.9). As a result we obtain an RGE for the phase variable $\boldsymbol{\varphi}$

$$\frac{\mathrm{d}\varphi_k(\theta)}{\mathrm{d}\theta} = \omega_{0k} - \epsilon^2 \sum_{\boldsymbol{m}>\boldsymbol{0}}{}' \sum_{\mu} \frac{m_l}{m_s\omega_{0s} + \mu\nu_{\boldsymbol{m}}} \frac{\partial^2 |V_{\boldsymbol{m}}(\boldsymbol{A};\mu)|^2}{\partial A_k \partial A_l}$$
$$+\epsilon^2 \sum_{\boldsymbol{m}>\boldsymbol{0}}{}' \sum_{\mu} \frac{m_l m_r}{(m_s\omega_{0s} + \mu\nu_{\boldsymbol{m}})^2} \frac{\partial}{\partial A_k} [\gamma_{lr}(\boldsymbol{A}) |V_{\boldsymbol{m}}(\boldsymbol{A};\mu)|^2]. \quad (3.14)$$

It is worthwhile to mention that the RGEs (3.13) and (3.14) are decoupled in the case when the system is far from a primary resonance of the form (3.5). This however does not hold in the resonance case, for which the first-order RGs comprise a coupled Hamiltonian system

$$\frac{\mathrm{d}\varphi_k(\theta)}{\mathrm{d}\theta} = \omega_{0k} + 2\epsilon \sum_{\boldsymbol{m}^{(\mathrm{R})}>\boldsymbol{0}} \frac{\partial}{\partial A_k} V_{\boldsymbol{m}^{(\mathrm{R})}} \left(\boldsymbol{A}; -\frac{m_s^{(\mathrm{R})}\omega_{0s}}{\nu_R} \right) \cos \boldsymbol{m}^{(\mathrm{R})} \cdot \boldsymbol{\varphi}, \quad (3.15)$$

$$\frac{\mathrm{d}A_k(\theta)}{\mathrm{d}\theta} = 2\epsilon \sum_{\boldsymbol{m}^{(\mathrm{R})}>\boldsymbol{0}} m_k^{(\mathrm{R})} V_{\boldsymbol{m}^{(\mathrm{R})}} \left(\boldsymbol{A}; -\frac{m_s^{(\mathrm{R})}\omega_{0s}}{\nu_R} \right) \sin \boldsymbol{m}^{(\mathrm{R})} \cdot \boldsymbol{\varphi}. \quad (3.16)$$

In writing equations (3.15) and (3.16) for simplicity we have assumed that the arguments of the Fourier amplitudes of the non-integrable part of the Hamiltonian (2.3) are equal to zero.

Equations (3.15) and (3.16) are the Hamilton equations of motion for an isolated resonance system, which have been obtained previously in the framework of the canonical perturbation theory [2, 7]. The RGEs (3.13) and (3.14) govern the long-timescale evolution of the amplitude-phase variables $(\boldsymbol{A}, \boldsymbol{\varphi})$ due to the nonlinear interaction between perturbation modes $V_{\boldsymbol{m}}(\boldsymbol{A};\mu)$. Similar equations can be derived by means of the multiple-scale method [5].

4. Renormalization group reduction of Liouville's equation

We consider the solution of equations (2.22) and (2.23) for small ϵ by means of the RG method. For that purpose we perform again the naive perturbation expansion

$$F = F^{(0)} + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + \cdots, \qquad G_{\boldsymbol{n}} = G_{\boldsymbol{n}}^{(0)} + \epsilon G_{\boldsymbol{n}}^{(1)} + \epsilon^2 G_{\boldsymbol{n}}^{(2)} + \cdots, \quad (4.1)$$

and substitute it into equations (2.22) and (2.23). The lowest-order perturbation equations have the obvious solution

$$F^{(0)} = F_0(\boldsymbol{J}), \qquad G^{(0)}_{\boldsymbol{n}} = W_{\boldsymbol{n}}(\boldsymbol{J}) \exp(-\mathrm{i} n_k \omega_{0k} \theta). \tag{4.2}$$

The first-order perturbation equations read as

$$\frac{\partial F^{(1)}}{\partial \theta} = \mathrm{i} \frac{\partial}{\partial J_k} \Big[\sum_{\boldsymbol{n}}{}' n_k V_{\boldsymbol{n}} W_{-\boldsymbol{n}} \exp(\mathrm{i} n_l \omega_{0l} \theta) \Big], \tag{4.3}$$

$$\begin{aligned} \frac{\partial G^{(1)}_{\boldsymbol{n}}}{\partial \theta} &= -\mathrm{i} n_k \omega_{0k} G^{(1)}_{\boldsymbol{n}} + \mathrm{i} n_k V_{\boldsymbol{n}} \frac{\partial F_0}{\partial J_k} \\ &\quad + \mathrm{i} \sum_{\boldsymbol{m}}{}' \Big[n_k V_{\boldsymbol{n}-\boldsymbol{m}} \frac{\partial W_{\boldsymbol{m}}}{\partial J_k} - m_k \frac{\partial}{\partial J_k} (V_{\boldsymbol{n}-\boldsymbol{m}} W_{\boldsymbol{m}}) \Big] \exp(-\mathrm{i} m_l \omega_{0l} \theta). \end{aligned} \tag{4.4}$$

We again assume that the modes $V_{\boldsymbol{n}}(\boldsymbol{J};\theta)$ are periodic in θ, so that they can be expanded in a Fourier series (3.4). If the original system (2.1) exhibits primary resonances of the form (3.5) in the case when ω_{0k} does not depend on the action variables, we can solve the first-order perturbation equations (4.3) and (4.4). The result is as follows:

$$\begin{aligned} F^{(1)} &= \mathrm{i} \mathcal{R}(\theta) \sum_{\boldsymbol{n}^{(\mathrm{R})}}{}'' n^{(\mathrm{R})}_k \frac{\partial}{\partial J_k} \Big[V_{\boldsymbol{n}^{(\mathrm{R})}} \Big(-\frac{n^{(\mathrm{R})}_l \omega_{0l}}{\nu_R} \Big) W_{-\boldsymbol{n}^{(\mathrm{R})}} \Big] \\ &\quad + \sum_{\boldsymbol{n}}{}' \sum_{\mu}{}' n_k \frac{\partial}{\partial J_k} [V_{\boldsymbol{n}}(\mu) W_{-\boldsymbol{n}}] \frac{\exp[\mathrm{i}(n_l \omega_{0l} + \mu \nu_{\boldsymbol{n}})\theta]}{n_l \omega_{0l} + \mu \nu_{\boldsymbol{n}}}, \end{aligned} \tag{4.5}$$

$$G^{(1)}_{\boldsymbol{n}} = \mathcal{G}_{\boldsymbol{n}} \exp(-\mathrm{i} n_k \omega_{0k} \theta), \tag{4.6}$$

where

$$\begin{aligned} \mathcal{G}_{\boldsymbol{n}} &= \mathrm{i} \mathcal{R}(\theta) \delta_{\boldsymbol{n}\boldsymbol{n}^{(\mathrm{R})}} n_k \frac{\partial F_0}{\partial J_k} V_{\boldsymbol{n}} \Big(-\frac{n^{(\mathrm{R})}_l \omega_{0l}}{\nu_R} \Big) + \mathrm{i} \mathcal{R}(\theta) \sum_{\boldsymbol{n}-\boldsymbol{n}^{(\mathrm{R})}}{}'' \Big\{ n_k V_{\boldsymbol{n}^{(\mathrm{R})}} \Big(-\frac{n^{(\mathrm{R})}_l \omega_{0l}}{\nu_R} \Big) \frac{\partial W_{\boldsymbol{n}-\boldsymbol{n}^{(\mathrm{R})}}}{\partial J_k} \\ &\quad - (n_k - n^{(\mathrm{R})}_k) \frac{\partial}{\partial J_k} \Big[V_{\boldsymbol{n}^{(\mathrm{R})}} \Big(-\frac{n^{(\mathrm{R})}_l \omega_{0l}}{\nu_R} \Big) W_{\boldsymbol{n}-\boldsymbol{n}^{(\mathrm{R})}} \Big] \Big\} \\ &\quad + n_k \frac{\partial F_0}{\partial J_k} \sum_{\mu}{}' V_{\boldsymbol{n}}(\mu) \frac{\exp[\mathrm{i}(n_l \omega_{0l} + \mu \nu_{\boldsymbol{n}})\theta]}{n_l \omega_{0l} + \mu \nu_{\boldsymbol{n}}} \\ &\quad + \sum_{\boldsymbol{m}}{}' \sum_{\mu}{}' \Big\{ n_k V_{\boldsymbol{n}-\boldsymbol{m}}(\mu) \frac{\partial W_{\boldsymbol{m}}}{\partial J_k} - m_k \frac{\partial}{\partial J_k} [V_{\boldsymbol{n}-\boldsymbol{m}}(\mu) W_{\boldsymbol{m}}] \Big\} \\ &\quad \times \frac{\exp\{\mathrm{i}[(n_l - m_l)\omega_{0l} + \mu \nu_{\boldsymbol{n}-\boldsymbol{m}}]\theta\}}{(n_l - m_l)\omega_{0l} + \mu \nu_{\boldsymbol{n}-\boldsymbol{m}}}. \end{aligned} \tag{4.7}$$

In the above expressions $\sum''$ denotes summation over all primary resonances (3.5). To obtain the desired RGEs we proceed in the same way as in the previous section. The first-order RGEs are

$$\frac{\partial F_0}{\partial \theta} = \mathrm{i} \epsilon \sum_{\boldsymbol{n}^{(\mathrm{R})}}{}'' n^{(\mathrm{R})}_k \frac{\partial}{\partial J_k} [V_{\boldsymbol{n}^{(\mathrm{R})}}(\boldsymbol{J};\mu_R) W_{-\boldsymbol{n}^{(\mathrm{R})}}(\boldsymbol{J})], \tag{4.8}$$

$$\begin{aligned} \frac{\partial W_{\boldsymbol{n}}}{\partial \theta} &= \mathrm{i} \epsilon \delta_{\boldsymbol{n}\boldsymbol{n}^{(\mathrm{R})}} n_k \frac{\partial F_0}{\partial J_k} V_{\boldsymbol{n}}(\boldsymbol{J};\mu_R) + \mathrm{i}\epsilon \sum_{\boldsymbol{n}-\boldsymbol{n}^{(\mathrm{R})}}{}'' \Big\{ n_k V_{\boldsymbol{n}^{(\mathrm{R})}}(\boldsymbol{J};\mu_R) \frac{\partial W_{\boldsymbol{n}-\boldsymbol{n}^{(\mathrm{R})}}}{\partial J_k} \\ &\quad - (n_k - n^{(\mathrm{R})}_k) \frac{\partial}{\partial J_k} [V_{\boldsymbol{n}^{(\mathrm{R})}}(\boldsymbol{J};\mu_R) W_{\boldsymbol{n}-\boldsymbol{n}^{(\mathrm{R})}}(\boldsymbol{J})] \Big\}, \end{aligned} \tag{4.9}$$

where

$$\mu_R = -\frac{n_k^{(\mathrm{R})}\omega_{0k}}{\nu_R}. \tag{4.10}$$

Equations (4.8) and (4.9) describe the resonant mode coupling when strong primary resonances are present in the original system.

Let us now assume that the original system is far from resonances. Solving the second-order perturbation equation for $F^{(2)}$ and $G_{\boldsymbol{n}}^{(2)}$

$$\frac{\partial F^{(2)}}{\partial \theta} = \mathrm{i}\frac{\partial}{\partial J_k}\Big(\sum_{\boldsymbol{n}}{}' n_k V_{\boldsymbol{n}} G_{-\boldsymbol{n}}^{(1)}\Big), \tag{4.11}$$

$$\frac{\partial G_{\boldsymbol{n}}^{(2)}}{\partial \theta} + \mathrm{i}n_k\omega_{0k}G_{\boldsymbol{n}}^{(2)} = \mathrm{i}n_k V_{\boldsymbol{n}}\frac{\partial F^{(1)}}{\partial J_k} + \mathrm{i}\sum_{\boldsymbol{m}}{}'\left[n_k V_{\boldsymbol{n}-\boldsymbol{m}}\frac{\partial G_{\boldsymbol{m}}^{(1)}}{\partial J_k} - m_k\frac{\partial}{\partial J_k}(V_{\boldsymbol{n}-\boldsymbol{m}}G_{\boldsymbol{m}}^{(1)})\right], \tag{4.12}$$

we obtain

$$F^{(2)} = 2\pi\mathcal{R}(\theta)\frac{\partial}{\partial J_k}\left[\sum_{\boldsymbol{n}>\boldsymbol{0}}{}'\sum_{\lambda} n_k n_l |V_{\boldsymbol{n}}(\lambda)|^{\boldsymbol{2}}\frac{\partial F_0}{\partial J_l}\Re(\gamma; n_s\omega_{0s} + \lambda\nu_{\boldsymbol{n}})\right] + \text{oscillating terms}, \tag{4.13}$$

$$G_{\boldsymbol{n}}^{(2)} = \mathcal{F}_{\boldsymbol{n}}\exp(-\mathrm{i}n_s\omega_{0s}\theta), \tag{4.14}$$

where

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{n}} = {} & \mathrm{i}\mathcal{R}(\theta)n_k n_l\sum_{\mu}\frac{V_{\boldsymbol{n}}(\mu)}{n_s\omega_{0s}+\mu\nu_{\boldsymbol{n}}}\frac{\partial^2}{\partial J_k\partial J_l}[V_{\boldsymbol{n}}^*(\mu)W_{\boldsymbol{n}}] \\
& + \mathrm{i}\mathcal{R}(\theta)\sum_{\boldsymbol{m}}{}'\sum_{\mu}\frac{1}{(n_s-m_s)\omega_{0s}+\mu\nu_{\boldsymbol{n}-\boldsymbol{m}}}\bigg\{-n_k m_l V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)\frac{\partial}{\partial J_k}\Big(V_{\boldsymbol{n}-\boldsymbol{m}}^*(\mu)\frac{\partial W_{\boldsymbol{n}}}{\partial J_l}\Big) \\
& + n_k n_l V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)\frac{\partial^2}{\partial J_k\partial J_l}(V_{\boldsymbol{n}-\boldsymbol{m}}^*(\mu)W_{\boldsymbol{n}}) + m_k m_l\frac{\partial}{\partial J_k}\Big(|V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)|^{\boldsymbol{2}}\frac{\partial W_{\boldsymbol{n}}}{\partial J_l}\Big) \\
& - m_k n_l\frac{\partial}{\partial J_k}\Big[V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)\frac{\partial}{\partial J_l}(V_{\boldsymbol{n}-\boldsymbol{m}}^*(\mu)W_{\boldsymbol{n}})\Big]\bigg\} + \text{oscillating terms},
\end{aligned} \tag{4.15}$$

and the functions $\mathcal{R}(\theta)$ and $\Re(x;y)$ are given by equations (3.11) and (3.12), respectively. It is now straightforward to write the second-order RGEs. They are

$$\frac{\partial F_0}{\partial \theta} = 2\pi\epsilon^2\frac{\partial}{\partial J_k}\left[\sum_{\boldsymbol{n}>\boldsymbol{0}}{}'\sum_{\lambda} n_k n_l |V_{\boldsymbol{n}}(\lambda)|^2\frac{\partial F_0}{\partial J_l}\Re(\gamma; n_s\omega_{0s} + \lambda\nu_{\boldsymbol{n}})\right], \tag{4.16}$$

$$\begin{aligned}
\frac{\partial W_{\boldsymbol{n}}}{\partial \theta} = {} & \mathrm{i}\epsilon^2 n_k n_l\sum_{\mu}\frac{V_{\boldsymbol{n}}(\mu)}{n_s\omega_{0s}+\mu\nu_{\boldsymbol{n}}}\frac{\partial^2}{\partial J_k\partial J_l}[V_{\boldsymbol{n}}^*(\mu)W_{\boldsymbol{n}}] \\
& + \mathrm{i}\epsilon^2\sum_{\boldsymbol{m}}{}'\sum_{\mu}\frac{1}{(n_s-m_s)\omega_{0s}+\mu\nu_{\boldsymbol{n}-\boldsymbol{m}}}\bigg\{-n_k m_l V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)\frac{\partial}{\partial J_k}\Big(V_{\boldsymbol{n}-\boldsymbol{m}}^*(\mu)\frac{\partial W_{\boldsymbol{n}}}{\partial J_l}\Big) \\
& + n_k n_l V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)\frac{\partial^2}{\partial J_k\partial J_l}(V_{\boldsymbol{n}-\boldsymbol{m}}^*(\mu)W_{\boldsymbol{n}}) + m_k m_l\frac{\partial}{\partial J_k}\Big(|V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)|^{\boldsymbol{2}}\frac{\partial W_{\boldsymbol{n}}}{\partial J_l}\Big) \\
& - m_k n_l\frac{\partial}{\partial J_k}\Big[V_{\boldsymbol{n}-\boldsymbol{m}}(\mu)\frac{\partial}{\partial J_l}(V_{\boldsymbol{n}-\boldsymbol{m}}^*(\mu)W_{\boldsymbol{n}})\Big]\bigg\}.
\end{aligned} \tag{4.17}$$

The RGE (4.16) is a Fokker–Planck equation describing the diffusion of the adiabatic action invariant. It has been derived previously by many authors (see e.g. [2] and references therein). It is important to note that our derivation does not require the initial assumption concerning the fast stochastization of the angle variable. The fact that the latter is indeed a stochastic variable is clearly visible from the second RGE (4.17), governing the slow amplitude evolution of the angle-dependent part of the phase space density. Although it looks complicated, its most important feature is that equations for the amplitudes of different modes are decoupled. In the case of isolated nonlinear resonance, equation (4.17) acquires a very simple form as will be shown in the next section.

5. Modulational diffusion

As an example to demonstrate the theory developed in previous sections, we consider the simplest example of a one-and-a-half-degree-of-freedom dynamical system exhibiting chaotic motion

$$H_0(J) = \lambda J + H_s(J), \qquad V(\alpha, J; \theta) = V(J)\cos(\alpha + \xi \sin \nu\theta). \tag{5.1}$$

The Hamiltonian (5.1), written in resonant canonical variables describes an isolated nonlinear resonance of one-dimensional betatron motion of particles in an accelerator with modulated resonant phase (or modulated linear betatron tune). The modulation may come from various sources: ripple in the power supply of quadrupole magnets, synchro-betatron coupling or ground motion. The resonance detuning λ defines the distance from the resonance, Ξ is the amplitude of modulation of the linear betatron tune and ν is the modulation frequency, where $\xi = \Xi/\nu$.

The modulation of the resonant phase (or the unperturbed tune) causes a weak instability induced by modulational layers. This phenomenon, usually referred to as modulational diffusion, has been studied extensively by many authors [20, 21] (see also [2] and references therein). Without loss of generality we consider ξ positive. Since

$$\omega_0 = \lambda + \frac{\mathrm{d}H_s}{\mathrm{d}J}, \qquad V_1(J; \mu) = \tfrac{1}{2} V(J)\mathcal{J}_\mu(\xi), \tag{5.2}$$

where $\mathcal{J}_n(z)$ is the Bessel function of order n, the RGE (3.13) for the amplitude A can be rewritten as

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{\pi\epsilon^2}{2\nu}\left\{\frac{\partial}{\partial A}[V^2(A)\mathcal{J}^2_{[\frac{\omega_0}{\nu}]}(\xi)] - \gamma(A)V^2(A)\frac{\partial}{\partial a}[\mathcal{J}^2_a(\xi)]\bigg|_{a=-\frac{\omega_0}{\nu}}\right\}. \tag{5.3}$$

Here the square brackets $[z]$ encountered in the index of the Bessel function imply integer part of z. Moreover, in deriving the expression for $V_1(J; \mu)$ in equation (5.2) use has been made of the identity

$$\exp(\mathrm{i}q\sin z) = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(|q|)\exp[\mathrm{i}nz\,\mathrm{sgn}(q)], \tag{5.4}$$

and finally, the limit $\gamma \longrightarrow 0$ in equation (3.13) has been taken. For small value of ξ utilizing the approximate expression for the derivative of Bessel functions with respect to the order we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{\pi\epsilon^2}{2\nu}\left\{\frac{\partial}{\partial A}[V^2(A)\mathcal{J}^2_{[\frac{\omega_0}{\nu}]}(\xi)] - 2\mathcal{J}^2_{[\frac{\omega_0}{\nu}]}(\xi)\ln\left(\frac{\xi}{2}\right)\gamma(A)V^2(A)\right\}. \tag{5.5}$$

Let us now turn to the RGEs (4.16) and (4.17). They can be rewritten in the form

$$\frac{\partial F_0}{\partial\theta} = \frac{\pi\epsilon^2}{2\nu}\frac{\partial}{\partial J}\left[V^2(J)\mathcal{J}^2_{[\frac{\omega_0}{\nu}]}(\xi)\frac{\partial F_0}{\partial J}\right], \tag{5.6}$$

$$\frac{1}{W_n}\frac{\partial W_n}{\partial\theta} = \frac{\mathrm{i}\pi\epsilon^2 n}{2\nu\sin(\pi\omega_0/\nu)}\mathcal{J}^2_{-\frac{\omega_0}{\nu}}(\xi)\frac{\mathrm{d}}{\mathrm{d}J}\left(V\frac{\mathrm{d}V}{\mathrm{d}J}\right) + \frac{\pi\epsilon^2 n}{2\nu}\mathcal{J}^2_{[\frac{\omega_0}{\nu}]}(\xi)\frac{\mathrm{d}}{\mathrm{d}J}\left(V\frac{\mathrm{d}V}{\mathrm{d}J}\right). \tag{5.7}$$

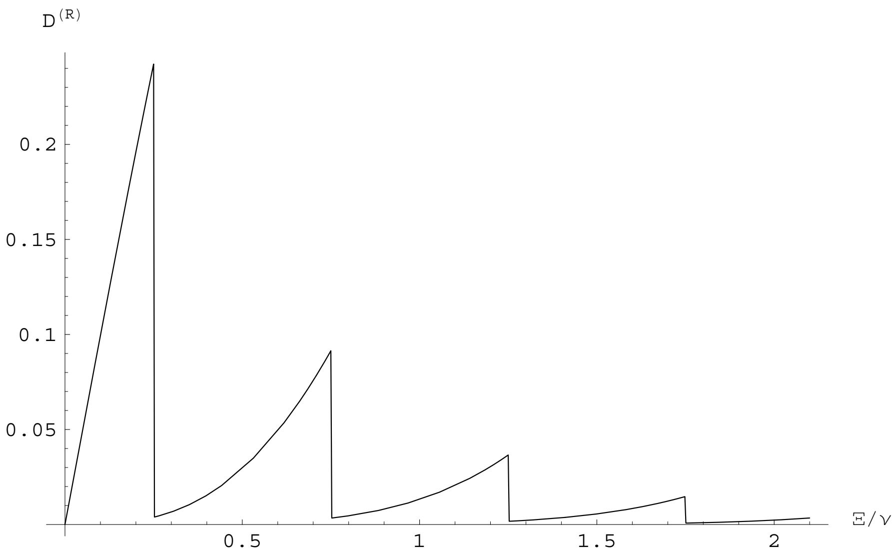

Figure 1. Reduced diffusion coefficient $\mathcal{D}^{(\mathrm{R})}$ as a function of the ratio $\xi = \Xi/\nu$ for $\lambda = 2\Xi$.

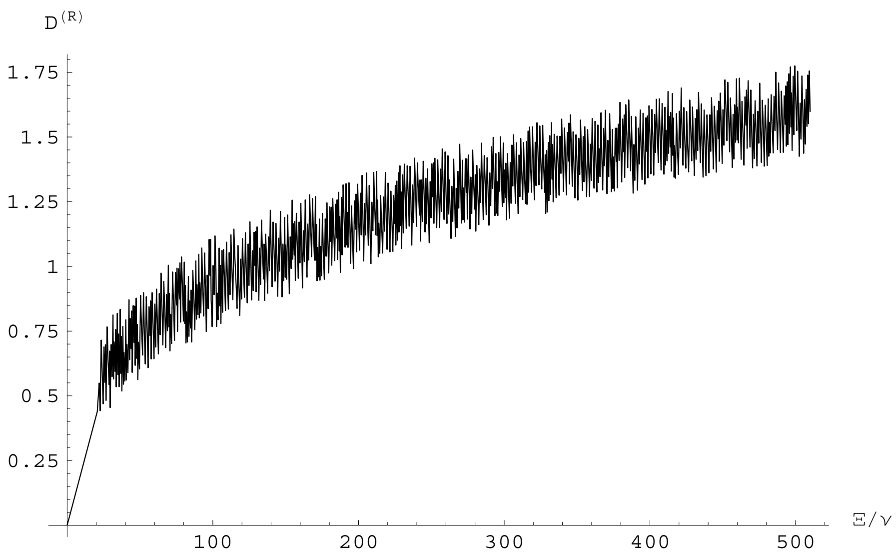

Figure 2. Reduced diffusion coefficient $\mathcal{D}^{(\mathrm{R})}$ as a function of the ratio $\xi = \Xi/\nu$ for $\lambda = \Xi$.

Equation (5.7) suggests that the amplitudes W_n of the angular modes G_n exhibit exponential growth with an increment

$$\Gamma = \frac{\pi\epsilon^2}{2\nu}\mathcal{J}^2_{[\frac{\omega_0}{\nu}]}(\xi)\frac{\mathrm{d}}{\mathrm{d}J}\left(V\frac{\mathrm{d}V}{\mathrm{d}J}\right). \tag{5.8}$$

Equation (5.6) is a Fokker–Planck equation for the angle-independent part of the phase space density with a diffusion coefficient

$$\mathcal{D}(J) = \frac{\pi\epsilon^2}{2\nu}V^2(J)\mathcal{J}_{[\frac{\omega_0}{\nu}]^2}(\xi). \tag{5.9}$$

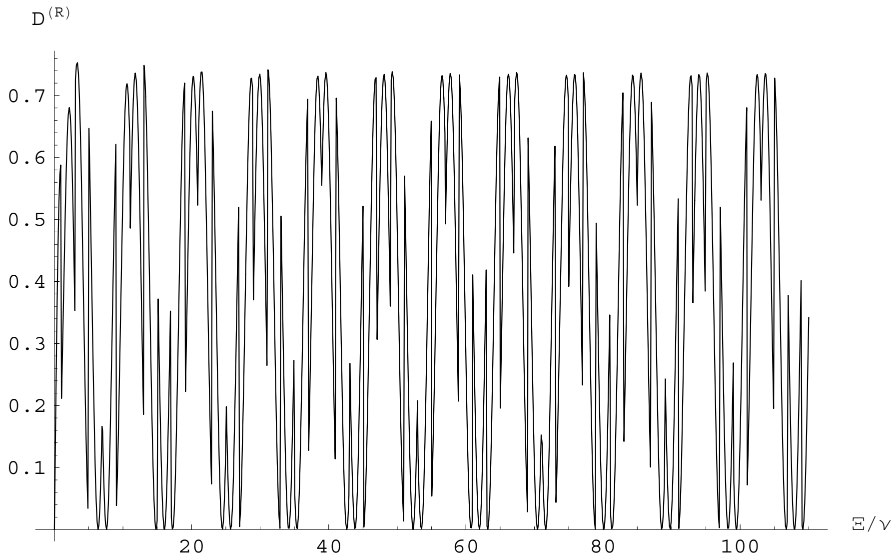

Figure 3. Reduced diffusion coefficient $\mathcal{D}^{(\mathrm{R})}$ as a function of the ratio $\xi = \Xi/\nu$ for $\lambda = \Xi/2$.

In figures 1–3 the reduced diffusion coefficient

$$\mathcal{D}^{(\mathrm{R})}\left(J, \frac{\Xi}{\nu}\right) = \frac{2\Xi}{\pi\epsilon^2 V^2(J)}\mathcal{D}(J) \tag{5.10}$$

has been plotted as a function of the ratio between the amplitude and the frequency of the modulation. Three typical regimes corresponding to different values of λ/Ξ used as a control parameter have been chosen. In the first one depicted in figure 1 we have taken the resonance detuning to be twice as large as the amplitude of the modulation ($\lambda = 2\Xi$). In this case there is no crossing of the main resonance described by the Hamiltonian (5.1) and the diffusion coefficient decreases very rapidly after reaching its maximum value at $\xi = 0.25$. The cases of periodic resonance crossings for $\lambda = \Xi$ and $\Xi/2$ are shown in figures 2 and 3, respectively.

6. Concluding remarks

In this paper we apply the RG method to the reduction of non-integrable multi-dimensional Hamiltonian systems. The notion of reduction is used here in the sense of slow, long-time behaviour, that survives appropriate averaging and/or factorizing of rapidly oscillating contributions to the dynamics of the system. It has been shown that the origin of the long-time relaxation effects in a nonlinear dynamical system is the resonant nonlinear interaction between perturbation modes, which causes the values of the adiabatic action invariants to diffuse away from any given region in phase space.

As a result of the investigation performed we have derived evolution equations for the slowly varying part of the angle-averaged phase space density, and for the amplitudes of the angular modes. It has been shown that these equations are the RGEs.

The case of a one-dimensional isolated nonlinear resonance with a resonant phase (or linear unperturbed tune) subjected to periodic modulation has been studied in detail. The coefficient of modulational diffusion, as well as the exponential growth increment of the amplitudes of angular modes, have been obtained in explicit form.

Acknowledgments

It is a pleasure to thank E Startsev for many useful discussions concerning the subject of the present paper. I am indebted to N Goldenfeld for careful reading of the manuscript, and for making valuable suggestions. I am also grateful to the unknown referee for the constructive and helpful criticism.

This work was supported by the US Department of Energy.

References

[1] Chirikov B V 1979 *Phys. Rep.* **52** 263

[2] Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (Berlin: Springer)

[3] Chattopadhyay S *et al* (ed) 1995 *Nonlinear Dynamics in Particle Accelerators: Theory and Experiments (Arcidosso, 1994) (AIP Conf. Proc. Vol. 344)* (New York: American Institute of Physics)

[4] Chattopadhyay S *et al* (ed) *Nonlinear and Collective Phenomena in Beam Physics (Arcidosso, 1996) (AIP Conf. Proc. Vol. 395)* (New York: American Institute of Physics)

[5] Nayfeh A H 1981 *Introduction to Perturbation Techniques* (New York: Wiley)

[6] Dragt A J 1982 *Physics of High Energy Particle Accelerators. Lectures on Nonlinear Orbit Dynamics (Fermilab Summer School, 1981) (AIP Conf. Proc. No 87)* ed R A Carrigan *et al* (New York: American Institute of Physics) pp 147–313

[7] Giacaglia G E O 1972 *Perturbation Methods in Nonlinear Systems (Appl. Math. Sci. No 8)* (Berlin: Springer)

[8] Cary J R 1981 *Phys. Rep.* **79** 129

[9] Chen L-Y, Goldenfeld N and Oono Y 1996 *Phys. Rev.* E **54** 376

[10] Nozaki K and Oono Y 2001 *Phys. Rev.* E **63** 046101-1

[11] Ei S-I, Fujii K and Kunihiro T 2000 *Ann. Phys., NY* **280** 236

[12] Goto S-I and Nozaki K 2001 *J. Phys. Soc. Japan* **70** 49

[13] Kunihiro T and Matsukidaira J 1998 *Phys. Rev.* E **57** 4817

[14] Arnold V I and Avez A 1968 *Ergodic Problems of Classical Mechanics* (New York: Benjamin)

[15] Zwanzig R W 1961 *Lectures in Theoretical Physics* vol 3, ed W E Brittin (New York: Wiley) pp 106–41

[16] Courant E D and Snyder H S 1958 *Ann. Phys., NY* **3** 1

[17] Tzenov S I *Contemporary Accelerator Physics* (Singapore: World Scientific) in preparation

[18] Sudbery A 1986 *Quantum Mechanics and the Particles of Nature* (Cambridge: Cambridge University Press)

[19] Zaslavsky G M 1985 *Chaos in Dynamical Systems* (Chur: Harwood Academic)

[20] Chirikov B V, Lieberman M A, Shepelyansky D L and Vivaldi F M 1985 *Physica* D **14** 289

[21] Vivaldi F M 1984 *Rev. Mod. Phys.* **56** 737