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The electromagnetic Darwin model for intense charged particle beams

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Abstract

The theoretical and numerical properties of the electromagnetic Darwin model for intense charged particle beams are investigated. The model neglects the transverse displacement current in Ampere's law and results in the elimination of high-frequency transverse electromagnetic waves and the associated retardation effects in the Vlasov–Maxwell equations. In this paper, two numerical schemes are presented for the purpose of circumventing the numerical instabilities associated with the presence of $\mathbf{E}^T[\equiv -(1/c)\partial\mathbf{A}/\partial t]$ in the equations of motion for particle codes, where $\mathbf{A}$ is the vector potential. The first relies on higher-order velocity moments for closure, and the other replaces the mechanical momentum, $\mathbf{p} = \gamma m\mathbf{v}$, by the canonical momentum, $\mathbf{P} = \mathbf{p} + (q/c)\mathbf{A}$, as the phase-space variable. The properties of these simulations schemes in the laboratory frame as well as in the beam frame are also discussed. These new numerical methods are most suitable for studying Weibel and two-stream instabilities in heavy ion fusion research.

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1. Introduction

The purpose of this paper is to understand the basic properties of the electromagnetic Darwin model for high intensity relativistic particle beams and for the development of efficient numerical particle simulation schemes. The model was originally proposed by Darwin to retain the lowest-order relativistic corrections through order v^2/c^2 [1] by neglecting the transverse induction current in Ampere's law. The net result is the elimination of light waves from the Maxwell–Vlasov system, which, in turn, greatly relaxes the time step restrictions for numerical simulations. Another interesting property of the model is that the resulting Maxwell's equations are now elliptic rather than full-blown wave equations, and the required numerical procedures for solving these equations are different and the

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time-step requirement is relaxed. One may argue that the Darwin model is also valid for highly relativistic beams, if retardation effects are not the important physics at hand. The use of the Darwin model in particle simulations of plasmas has a long history. However, as noted in the past [2], the presence of the time derivative of the vector potential, $\partial \mathbf{A}/\partial t$, in the equations of motion for the Darwin model can cause numerical instabilities. To circumvent this difficulty, procedures involving the removal of $\partial \mathbf{A}/\partial t$ in the equations of motion have been developed, and the Darwin model has been successfully used in particle simulations for studying electromagnetic perturbations in plasmas, such as Weibel instabilities [2,3], whistler and magnetosonic waves [4], shear-Alfvén waves [5] as well as other applications [6,7]. In this paper, we adopt two different procedures for studying high-intensity particle beams. The first uses higher-order velocity moments to get rid of the troublesome time derivative of the vector potential similar to the method used for shear-Alfven waves [8], and the second uses a procedure similar to the Hamiltonian formulation suggested by Nielson and Lewis [2] in which the mechanical momentum, $\mathbf{p} = \gamma m\mathbf{v}$, is replaced by the canonical momentum, $\mathbf{P} = \mathbf{p} + (q/c)\mathbf{A}$, as a phase-space variable so as to eliminate the troublesome $\partial \mathbf{A}/\partial t$ term, where q is the charge, c is the speed of light in vacuo, and m is the rest mass. The advantage of using the canonical momentum has long been recognized in magnetic fusion research for test particle transport [9] and for gyrokinetic theory [10,11]. The present schemes can easily be cast into the $\delta f (\equiv F - F_0)$ formalism [12,13], where F is the distribution function in phase space, and F_0 is the equilibrium distribution. As a result, the simulation plasma has minimal numerical noise, and also provides us with the ability to easily access both linear and nonlinear regimes for the physics of interest. Since the high-frequency waves associated with the radiation fields are absent from the simulations, we can use the schemes of adiabatic particle pushing [14] for which the electrons are advanced more often, and with smaller time steps, than those for the ions and field equations so as to compensate for the mass ratio disparities for different charge species. These schemes are ideal for studying two-stream [15], filamentation [16] and Weibel [17] instabilities, which may cause deterioration of the beam quality in the heavy ion fusion driver and fusion chamber. The paper is organized as follows. In Section 2, the basic formalisms and properties for the intense beams based on the Darwin model are given. The basic schemes using higher-order velocity moments are discussed in Section 3, and the algorithms utilizing canonical momenta are presented in Section 4. In Section 5, the associated nonlinear δf formalisms are briefly described and concluding remarks are presented in Section 6.

2. Darwin model for relativistic beams

The relativistic form of the Vlasov equation describing the propagation of an intense particle beam with narrow momentum spread through a smooth-focusing transverse external focusing field can be expressed as [14]

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{x}} + \left[-\gamma_b m\omega_\beta^2 \mathbf{x}_\perp + q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\right] \times \frac{\partial F}{\partial \mathbf{p}} = 0 \tag{1}$$

where $\mathbf{v} = \mathbf{p}/\gamma m$, $\gamma_b = 1/\sqrt{1-\beta_b^2}$ and $\beta_b = \langle v_z\rangle/c$, $\langle v_z\rangle$ is the average axial velocity, $\mathbf{x}_\perp$ is the perpendicular displacement, z is the direction of beam propagation, and $\omega_\beta = const.$ is the effective applied betatron frequency for transverse (perpendicular) oscillations. In terms of longitudinal (L) and transverse (T) quantities relative to the direction of wave propagation, the reduced Maxwell's equations for the Darwin model can be expressed as

$$\nabla\cdot\mathbf{E}^L = 4\pi\rho \tag{2}$$

$$\nabla\times\mathbf{B} = (4\pi/c)\mathbf{J}^T \tag{3}$$

$$\nabla\times\mathbf{E}^T = -(1/c)\partial\mathbf{B}/\partial t \tag{4}$$

and $\nabla\cdot\mathbf{B} = 0$, where $\rho = q\int F\,\mathrm{d}\mathbf{p}$, $\mathbf{J} = q\int \mathbf{v}F\,\mathrm{d}\mathbf{p}$, $\mathbf{E}^L = -\nabla\Phi$,

$$\mathbf{J}^T = \mathbf{J} - \frac{1}{4\pi}\frac{\partial\nabla\Phi}{\partial t} \tag{5}$$

and $\nabla \times \mathbf{E}^L = 0$, $\nabla \cdot \mathbf{E}^T = 0$ and $\nabla \cdot \mathbf{J}^T = 0$. The only difference from the original Maxwell's equations is that the transverse induction current, $(1/c)\partial\mathbf{E}^T/\partial t$, is neglected on the right-hand side of Eq. (3). Letting $\mathbf{B} = \nabla \times \mathbf{A}$ and using the Coulomb gauge with $\nabla \cdot \mathbf{A} = 0$, Eqs. (3) and (4) become $\nabla^2\mathbf{A} = -(4\pi/c)\mathbf{J}^T$ and

$$\mathbf{E}^T = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{6}$$

respectively. For comparison, the original Ampere's law takes the form of $\nabla^2\mathbf{A} - (1/c^2)\partial^2\mathbf{A}/\partial t^2 = -(4\pi/c)\mathbf{J}^T$. Thus, we can view the Darwin model simply as the low-frequency version of the original Maxwell's equations valid for $\omega^2 \ll k^2c^2$, where ω and $\mathbf{k}$ are the characteristic frequency and wavenumbers, respectively, for the perturbations proportional to $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$. The boundary conditions for these equations are problem dependent. We can now write the Vlasov equation (1) for the relativistic beam in terms of the new variables as

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{x}} + \left\{ -\gamma_b m\omega_\beta^2 \mathbf{x}_\perp - q\left(\nabla\Phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) + \frac{q}{c}\left[\mathbf{v}_\perp \times \nabla_\perp \times \mathbf{A}_\perp + \frac{\partial \mathbf{v}_\perp\cdot\mathbf{A}_\perp}{\partial z}\hat{\mathbf{z}} - v_z\frac{\partial \mathbf{A}_\perp}{\partial z} + v_z\nabla_\perp A_z - (\mathbf{v}_\perp\cdot\nabla_\perp)A_z\hat{\mathbf{z}}\right]\right\}\frac{\partial F}{\partial \mathbf{p}} = 0. \tag{7}$$

The associated Poisson's equation and Ampere's law are

$$\nabla^2\Phi = -4\pi q\int F\,\mathrm{d}\mathbf{p} \tag{8}$$

and

$$\nabla^2\mathbf{A} = -\frac{4\pi q}{c}\int \mathbf{v}F\,\mathrm{d}\mathbf{p} + \frac{1}{c}\frac{\partial\nabla\Phi}{\partial t} \tag{9}$$

respectively, where $\mathbf{v}_\perp \times \nabla_\perp \times \mathbf{A}_\perp = (\partial A_y/\partial x - \partial A_x/\partial y)(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}})$. This system of [Eqs. (7)–(9)] for the relativistic beam is energy conserving, i.e.,

$$\frac{\partial}{\partial t}\left\langle \int \frac{\mathbf{p}\cdot\mathbf{p}}{2\gamma m}F\,\mathrm{d}\mathbf{p} + \frac{\gamma_b m\omega_\beta^2}{2}\mathbf{x}_\perp\cdot\mathbf{x}_\perp\int F\,\mathrm{d}\mathbf{p} + \frac{1}{8\pi}[|\nabla\Phi|^2 + |\nabla A_x|^2 + |\nabla A_y|^2 + |\nabla A_z|^2]\right\rangle_{\mathbf{x}} = 0 \tag{10}$$

where $\langle\cdots\rangle_{\mathbf{x}} = \int \mathrm{d}\mathbf{x}/V$ denotes spatial average. Letting $v_z \approx c\beta_b$, we find from Eqs. (8) and (9) that

$$A_z \approx \beta_b\left(1 - \frac{\omega}{k_z v_z}\frac{k_z^2}{k^2}\right)\Phi.$$

Using the approximation

$$\left|\frac{\omega}{k_z v_z}\frac{k_z^2}{k^2}\right| \ll 1 \tag{11}$$

we approximate Ampere's law, Eq. (9), by

$$\nabla^2\mathbf{A} = -\frac{4\pi q}{c}\int \mathbf{v}F\,\mathrm{d}\mathbf{p} \tag{12}$$

where $k^2 = k_\perp^2 + k_z^2$ and the $\mathbf{A}_\perp$ part of the equation is obtained through the use of $\nabla\cdot\mathbf{A} = 0$. Thus, Eqs. (7), (8) and (12) form the basic equations for the low-frequency ($\omega/k_z v_z \ll 1$) and long-thin ($k_z^2/k^2 \ll 1$) approximations of intense beams. The low-frequency condition in Eq. (11) is more stringent than the condition $\omega \ll kc$ used in the Darwin model. The approximation on the transverse current can also be understood from Eq. (5) by using $\nabla\cdot\mathbf{J}^T = 0$ to obtain

$$\mathbf{J}^T = \mathbf{J} - \frac{\mathbf{k}\cdot\mathbf{J}}{k^2}\mathbf{k} \tag{13}$$

in Fourier $\mathbf{k}$-space. For $J_z \gg J_\perp$, the transverse current becomes

$$J_z^T = J_z\left(1 - \frac{k_z^2}{k^2}\right), \quad \mathbf{J}_\perp^T = -\frac{k_z J_z}{k^2}\mathbf{k}_\perp.$$

With the additional long-thin approximation of $k_z \ll k$, we can further assume that $\mathbf{A}_\perp \approx 0$, $J_z^T \approx J_z$, and $\mathbf{J}_\perp^T \approx 0$. The corresponding Ampere's law is simplified to become

$$\nabla^2 A_z = -\frac{4\pi q}{c}\int v_z F\,\mathrm{d}\mathbf{p}. \tag{14}$$

Eq. (14) together with Poisson's equation (8) and the simplified version of Eq. (7), i.e.,

$$\frac{\partial F}{\partial t}+\mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{x}}+\left[-\gamma_b m\omega_\beta^2\mathbf{x}_\perp-q\left(\nabla\Phi+\frac{1}{c}\frac{\partial A_z}{\partial t}\hat{\mathbf{z}}\right)+\frac{q}{c}v_z\nabla_\perp A_z-\frac{q}{c}(\mathbf{v}_\perp\cdot\nabla_\perp)A_z\hat{\mathbf{z}}\right]\cdot\frac{\partial F}{\partial \mathbf{p}}=0 \tag{15}$$

form the governing equations for a long-thin relativistic beam. This set of equations conserves energy which takes the form of Eq. (10) minus the $A_\perp$ terms. For comparison, the present long-thin Darwin model is similar to the electrostatic/magnetostatic model used in previous studies [14], for which only the term of $(q/c)v_z\nabla_\perp A_z$ associated with the vector potential is kept in the Vlasov equation.

3. Darwin model I for particle simulation

The governing equations for a general relativistic beam are Eqs. (7)–(9). To circumvent the numerical difficulty, let us first follow the procedures similar to those developed for shear-Alfvén waves in gyrokinetic plasmas [8]. The associated characteristics for Eq. (7) are

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=\frac{\mathbf{p}}{\gamma m} \tag{16}$$

and

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}= & -\gamma_b m\omega_\beta^2\mathbf{x}_\perp-q\left(\nabla\Phi+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)\\ & +\frac{q}{c}\left[\mathbf{v}_\perp\times\nabla_\perp\times\mathbf{A}_\perp+\frac{\partial\mathbf{v}_\perp\cdot\mathbf{A}_\perp}{\partial z}\hat{\mathbf{z}}\right.\\ & \left.-v_z\frac{\partial\mathbf{A}_\perp}{\partial z}+v_z\nabla_\perp A_z-(\mathbf{v}_\perp\cdot\nabla_\perp)A_z\hat{\mathbf{z}}\right]\end{aligned} \tag{17}$$

and they are related to Eq. (7) through the Klimontovich–Dupree representation

$$f(\mathbf{x},\mathbf{p},t)=\sum_{j=1}^{N}\delta[\mathbf{x}-\mathbf{x}_j(t)]\delta[\mathbf{p}-\mathbf{p}_j(t)]$$

where N is the total number of simulation particles. The equation of motion $\mathrm{d}\mathbf{p}/\mathrm{d}t=-(q/c)\partial\mathbf{A}/\partial t+\cdots$ can cause numerical difficulties in particle simulations [2] because of the time-centering problem in particle pushing. These difficulties are avoidable if we calculate the time derivatives directly from the higher-order velocity moments. Specifically, let us first take the time derivatives of Eq. (9) and substitute the resulting expression for $\partial F/\partial t$ from Eq. (7). We then obtain

$$\nabla^2\frac{\partial\mathbf{A}}{\partial t}=-\frac{4\pi q}{c}\int\mathbf{v}\frac{\partial F}{\partial t}\,\mathrm{d}\mathbf{p}+\frac{1}{c}\nabla\frac{\partial^2\Phi}{\partial t^2} \tag{18}$$

where

$$\begin{aligned}\int\mathbf{v}\frac{\partial F}{\partial t}\,\mathrm{d}\mathbf{p}= & -\nabla\cdot\int\mathbf{v}\mathbf{v}F\,\mathrm{d}\mathbf{p}-\left[\gamma_b\frac{\omega_\beta^2}{\gamma}\mathbf{x}_\perp\right.\\ & \left.+\frac{q}{\gamma m}\left(\nabla\Phi+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)\right]\int F\,\mathrm{d}\mathbf{p}\\ & -\frac{q}{c}\frac{1}{\gamma m}\left[(\nabla_\perp\times\mathbf{A}_\perp)\times\int\mathbf{v}_\perp F\,\mathrm{d}\mathbf{p}\right.\\ & -\hat{\mathbf{z}}\frac{\partial}{\partial z}\mathbf{A}_\perp\cdot\int\mathbf{v}_\perp F\,\mathrm{d}\mathbf{p}\\ & +\left(\frac{\partial\mathbf{A}_\perp}{\partial z}-\nabla_\perp A_z\right)\int v_z F\,\mathrm{d}\mathbf{p}\\ & \left.+\hat{\mathbf{z}}\left(\nabla_\perp A_z-\frac{\partial\mathbf{A}_\perp}{\partial z}\right)\cdot\int\mathbf{v}_\perp F\,\mathrm{d}\mathbf{p}\right].\end{aligned}$$

Thus, in addition to the equations of motion, Eqs. (16) and (17), and the governing field equations, Eqs. (8) and (9), we need to solve an additional equation, Eq. (18), for $\partial\mathbf{A}/\partial t$ explicitly from higher-order velocity moments, together with

$$\nabla^2\frac{\partial^2\Phi}{\partial t^2}=4\pi q\nabla\cdot\int\mathbf{v}\frac{\partial F}{\partial t}\,\mathrm{d}\mathbf{p}$$

and

$$\nabla^2\frac{\partial\Phi}{\partial t}=4\pi q\nabla\cdot\int\mathbf{v}F\,\mathrm{d}\mathbf{p}$$

for the time derivatives in Eqs. (18) and (8), respectively, based on similar procedures. For the low-frequency approximation with $\omega\ll k_z v_z$, the governing equations become Eqs. (8), (12), (16), (17) and

$$\nabla^2\frac{\partial\mathbf{A}}{\partial t}=-\frac{4\pi q}{c}\int\mathbf{v}\frac{\partial F}{\partial t}\,\mathrm{d}\mathbf{p} \tag{19}$$

by neglecting the time derivatives of Φ. On the other hand, for a long-thin relativistic beam, the governing equations from Eq. (15) are:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=\frac{\mathbf{p}}{\gamma m}$$

and

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}= -\gamma_b m\omega_\beta^2\mathbf{x}_\perp - q\left(\nabla\Phi+\frac{1}{c}\frac{\partial A_z}{\partial t}\right) + \frac{q}{c}[v_z\nabla_\perp A_z-(\mathbf{v}_\perp\cdot\nabla_\perp)A_z\hat{\mathbf{z}}].$$

The associated field quantities are given by Eqs. (8) and (14). Again, the $\partial A_z/\partial t$ term can be evaluated directly from Eq. (14) as

$$\left[\nabla^2-\frac{4\pi q^2}{\gamma mc^2}\int F\,\mathrm{d}\mathbf{p}\right]\frac{\partial A_z}{\partial t} = \frac{4\pi q}{c}\left[\nabla\cdot\int \mathbf{v}v_zF\,\mathrm{d}\mathbf{p}+\frac{q}{\gamma m}\frac{\partial\Phi}{\partial z}\int F\,\mathrm{d}\mathbf{p} + \frac{q}{\gamma m}\nabla_\perp A_z\cdot\int \mathbf{v}_\perp F\,\mathrm{d}\mathbf{p}\right] \tag{20}$$

to circumvent the numerical instabilities. In this section, we have outlined the simple procedure for simulating high-intensity relativistic beams in the laboratory frame. The formalism presented here can easily be generalized to multi-species beam-plasma systems.

4. Darwin model II for particle simulations

Here, let us introduce a different way to eliminate the time derivatives of the vector potential by introducing the canonical momentum

$$\mathbf{P}\equiv\mathbf{p}+\frac{q}{c}\mathbf{A} \tag{21}$$

as a new phase-space variable. From

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\equiv\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\cdot\frac{\partial\mathbf{A}}{\partial\mathbf{x}}$$

we rewrite the orbit characteristics as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=\frac{\mathbf{p}}{\gamma m} \tag{22}$$

and

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}=-\gamma_b m\omega_\beta^2\mathbf{x}_\perp-q\nabla\Phi+\frac{q}{c}\nabla(\mathbf{v}\cdot\mathbf{A}). \tag{23}$$

Thus, by transforming from $\mathbf{p}$ to $\mathbf{P}$, the time derivative of $\mathbf{A}$ conveniently disappears from the equations of motion. The Vlasov equation in the new variables can be expressed as

$$\frac{\mathrm{d}F}{\mathrm{d}t}\equiv\frac{\partial F}{\partial t}+\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial F}{\partial\mathbf{x}}+\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\cdot\frac{\partial F}{\partial\mathbf{P}}=0 \tag{24}$$

where the characteristics are defined by Eqs. (22) and (23). The corresponding form of Poisson's equation is

$$\nabla^2\Phi=-4\pi q\int F\,\mathrm{d}\mathbf{P} \tag{25}$$

and Ampere's law can be expressed as

$$\left(\nabla^2-\frac{\omega_p^2}{c^2}\right)\mathbf{A}=-\frac{4\pi}{c}q\int\frac{\mathbf{P}}{\gamma m}F\,\mathrm{d}\mathbf{P}+\frac{1}{c}\nabla\frac{\partial\Phi}{\partial t} \tag{26}$$

where $\omega_p\equiv\sqrt{(4\pi nq^2/m)\int F\,\mathrm{d}\mathbf{P}/\gamma}$ is the relativistic plasma frequency. As before, the $\partial\Phi/\partial t$ term can then be calculated by the combination of Poisson's equation and the continuity equation, i.e.,

$$\nabla^2\frac{1}{c}\frac{\partial\Phi}{\partial t}+\nabla\cdot\left(\frac{\omega_p^2}{c^2}\mathbf{A}\right) = 4\pi q\nabla\cdot\left(\int\frac{\mathbf{P}}{\gamma mc}F\,\mathrm{d}\mathbf{P}\right). \tag{27}$$

Thus, Eqs. (22)–(27) constitute the Darwin model for an intense relativistic beam. Computationally, this formalism seems to be much simpler. However, the calculations of $\mathbf{P}(\mathbf{x},\mathbf{v})$ may cause problems, if $|q\mathbf{A}/c|\gg|\mathbf{p}|$. For the low frequency approximation with $\omega/k_zv_z\ll 1$, Eqs. (22)–(26) are the governing equations, in which the term, $\nabla\partial\Phi/\partial t$, in Eq. (26) is negligible. For the long, thin-beam approximation, we can again let $\mathbf{A}_\perp\approx 0$, so that

$$\mathbf{P}\equiv\mathbf{p}+\frac{q}{c}A_z\hat{\mathbf{z}}.$$

Thus, the corresponding governing equations are now Eqs. (22)–(26) without the terms associated with $\mathbf{A}_\perp$ and $\partial\Phi/\partial t$.

For a one-component relativistic beam traveling with a constant average velocity of $v_z\approx\beta_b c$, which gives $A_z\simeq\beta_b\Phi$, the Darwin model in the laboratory frame becomes

$$\frac{\partial F}{\partial t}+\mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{x}}+\left[-\gamma_b m\omega_\beta^2\mathbf{x}_\perp-\frac{q}{\gamma_b^2}\nabla\Phi+\frac{q}{c}\nabla(\mathbf{v}_\perp\cdot\mathbf{A}_\perp)\right]\cdot\frac{\partial F}{\partial \mathbf{P}}=0 \tag{28}$$

where $\mathbf{P}=\mathbf{p}+(q/c)(\beta_b\Phi\hat{\mathbf{z}}+\mathbf{A}_\perp)$, with

$$\nabla^2\Phi=-4\pi q\int F\,\mathrm{d}\mathbf{P} \tag{29}$$

and

$$\nabla^2\mathbf{A}_\perp=-\frac{4\pi q}{c}\int\mathbf{v}_\perp F\,\mathrm{d}\mathbf{P}. \tag{30}$$

We then obtain the single-beam Darwin model *in the beam frame* ('primed' variables) by setting $\gamma_b=1$, and $\beta_b=0$ and replacing $(\mathbf{x},\mathbf{P},t)\to(\mathbf{x}',\mathbf{P}',t')$. For $\mathbf{A}_\perp\approx 0$, we recover the usual electrostatic model *in the beam frame* [18]. Thus, we present here a different approach to derive the same model. Similarly, we can obtain the one-component Vlasov equation in the beam frame in $\mathbf{p}'$ coordinates from Eq. (28) as

$$\frac{\partial F}{\partial t'}+\mathbf{v}'\cdot\frac{\partial F}{\partial \mathbf{x}'}-\left[m\omega_\beta^2\mathbf{x}'_\perp+q\nabla'\Phi+\frac{q}{c}\times\left(\frac{\partial\mathbf{A}_\perp}{\partial t'}-\mathbf{v}'_\perp\times\nabla'_\perp\times\mathbf{A}_\perp-\hat{\mathbf{z}}'\frac{\partial}{\partial z'}(\mathbf{v}'_\perp\cdot\mathbf{A}_\perp)+v'_z\frac{\partial\mathbf{A}_\perp}{\partial z'}\right)\right]\cdot\frac{\partial F}{\partial\mathbf{p}'}=0$$

and the potentials are given by Eqs. (29) and (30) with $\nabla\to\nabla'$, $\mathbf{v}_\perp\to\mathbf{v}'_\perp$ and $\mathbf{P}\to\mathbf{p}'$. This system of equations in the beam frame conserves energy similar to that of Eq. (10).

5. Nonlinear δf formalism

The corresponding δf formalism [12–14] in the laboratory frame for the Darwin model can be derived by expressing $F=F_0+\delta f$, $\Phi=\Phi_0+\delta\Phi$ and $\mathbf{A}=\mathbf{A}_0+\delta\mathbf{A}$, where F_0 satisfies

$$\frac{\partial F_0}{\partial t}+\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial F_0}{\partial\mathbf{x}}+\left.\frac{\mathrm{d}(\mathbf{p},\mathbf{P})}{\mathrm{d}t}\right|_0\cdot\frac{\partial F_0}{\partial(\mathbf{p},\mathbf{P})}=0$$

and $|_0$ denotes the zeroth-order trajectories calculated by using the equilibrium potentials, Φ_0 and $\mathbf{A}_0$. The perturbed distribution is determined from

$$\frac{\mathrm{d}\delta f}{\mathrm{d}t}=-\left.\frac{\mathrm{d}(\mathbf{p},\mathbf{P})}{\mathrm{d}t}\right|_\delta\cdot\frac{\partial F_0}{\partial(\mathbf{p},\mathbf{P})} \tag{31}$$

where $|_\delta$ denotes the perturbed trajectories obtained by using the perturbed potentials, $\delta\Phi$ and $\delta\mathbf{A}$. Defining $w=\delta f/F$, the weight function evolves according to

$$\frac{\mathrm{d}w}{\mathrm{d}t}=(1-w)\frac{1}{F_0}\frac{\mathrm{d}\delta f}{\mathrm{d}t}. \tag{32}$$

In the Klimontovich–Dupree representation, the perturbed distribution is related to the particle weight through

$$\delta f=\sum_{j=1}^{N}w_j\delta[\mathbf{x}-\mathbf{x}_j]\delta[(\mathbf{p},\mathbf{P})-(\mathbf{p}_j,\mathbf{P}_j)] \tag{33}$$

where N is the total number of particles in the simulation. The time evolution of $\mathbf{x}_j$, $\mathbf{p}_j$, $\mathbf{P}_j$, and w_j for the jth particle are described by the appropriate equations of motion discussed in Sections 3 and 4. For the field equations, the zeroth-order potentials, Φ_0 and $\mathbf{A}_0$, are obtained by using F_0, and the perturbed potentials, $\delta\Phi$ and $\delta\mathbf{A}$, can be obtained by using δf.

6. Conclusions

The non-radiative Darwin models developed here for particle simulations are similar to the usual electro-magnetostatic model [14] and represent an improvement over the model described earlier [19] by including all three components of **P** and **A** in the field equations. With the absence of the high-frequency radiation in the model, it has many numerical advantages over fully electromagnetic codes. The application of this model to the study of high-intensity beams will be reported elsewhere.

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References

[1] C.G. Darwin, Philos. Mag. 39 (1920) 537.
[2] C.W. Nielson, H.R. Lewis, Methods in Computational Physics, vol. 16, Academic Press, New York, 1976, p. 976.
[3] R.C. Davidson, D.A. Hammer, I. Haber, C.E. Wagner, Phys. Fluids 15 (1972) 317.
[4] J. Busnardo-Neto, P.L. Pritchett, A.T. Lin, J.M. Dawson, J. Comput. Phys. 23 (1977) 300.
[5] W.W. Lee, M.S. Chance, H. Okuda, Phys. Rev. Lett. 46 (1981) 1675.
[6] D.W. Hewett, J.K. Boyd, J. Comput. Phys. 70 (1987) 166.
[7] Sonnendrucker, et al., J. Comput. Phys. 121 (1995) 218.
[8] W.W. Lee, J.L.V. Lewandowski, T.S. Hahm, Z. Lin, Phys. Plasmas 8 (2001) 4435.
[9] R.B. White, M.S. Chance, Phys. Fluids 27 (1984) 2455.
[10] R.G. Littlejohn, Phys. Fluids 27 (1984) 976.
[11] T.S. Hahm, W.W. Lee, A. Brizard, Phys. Fluids 31 (1987) 1940.
[12] S.E. Parker, W.W. Lee, Phys. Fluids B 5 (1993) 77.
[13] W.W. Lee, Q. Qian, R.C. Davidson, Phys. Lett. A 230 (1997) 347.
[14] H. Qin, R.C. Davidson, W.W. Lee, Phys. Rev. Special Topics on Accelerators and Beams 3 (2000) 084401;
H. Qin, R.C. Davidson, W.W. Lee, Phys. Rev. Special Topics on Accelerators and Beams 3 (2000) 109901.
[15] H. Qin, R.C. Davidson, E.A. Startsev, Phys. Rev. Special Topics AB 6 (2003) 014401.
[16] E.P. Lee, S. Yu, H.L. Buchanan, F.W. Chambers, M.N. Rosenbluth, Phys. Fluids 23 (1980) 2095.
[17] E.A. Startsev, R.C. Davidson, Phys. Plasmas 12 (2003) 4928.
[18] R.C. Davidson, W.W. Lee, H. Qin, E. Startsev, Phys. Plasmas 9 (2002) 340.
[19] W.W. Lee, H. Qin, R.C. Davidson, Nucl. Instr. and Meth. A 464 (2001) 465.