SURVEY OF COLLECTIVE INSTABILITIES AND BEAM-PLASMA INTERACTIONS IN INTENSE HEAVY ION BEAMS*

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Nonlinear Kinetic Stability Theorem

- An important feature of intense beam propagation is the existence of a stability theorem based on the nonlinear Vlasov-Maxwell equations.*

- For a long, one-component coasting beam in the smooth-focusing approximation, the stability theorem expressed in the beam frame ($\beta_b = 0$ and $\gamma_b = 1$) states that any equilibrium distribution function $f_b^0(H)$ that satisfies
$$\frac{\partial}{\partial H} f_b^0(H) \leq 0$$
is nonlinearly stable to perturbations with arbitrary polarization.

- Here, $H = (p_r^2 + p_\theta^2 + p_z^2)/2m_b + m_b\omega_f^2 r^2/2 + e_b\phi^0(r)$ is the single-particle Hamiltonian in the beam frame, and $\phi^0(r)$ is the electrostatic potential determined self-consistently from Poisson's equation.

- A necessary condition for instability is that the beam distribution function have some nonthermal feature such as an inverted population in phase space, or a strong energy anisotropy, or that the beam have directed kinetic energy relative to background charge components.

*R. C. Davidson, Phys. Rev. Lett. **81**, 991 (1998).

Collective Instabilities in Intense Charged Particle Beams

One-Component Beams

- Electrostatic Harris instability ($T_{\parallel b}/T_{\perp b} < 1$)
- Electromagnetic Weibel instability ($T_{\parallel b}/T_{\perp b} < 1$)

Propagation Through Background Electrons

- Electron-ion two-stream (Electron Cloud) instability

Propagation Through Background Plasma

- Resistive hose instability
- Multispecies Weibel instability
- Multispecies two-stream instability

Harris Instability in Intense One-Component Beams

- Electrostatic Harris instability* can play an important role in multispecies plasmas with temperature anisotropy $T_{\|j} < T_{\perp j}$.

- Harris instability is inherently three-dimensional and involves a coupling of the longitudinal and transverse particle dynamics.

- Harris-like instability† also exists in intense one-component beams provided the anisotropy is sufficiently large ($T_{\|b}/T_{\perp b} \ll 1$) and the beam intensity is sufficiently large.

*E. G. Harris, Phys. Rev. Lett. **2**, 34 (1959).

†I. Haber, et al., Phys. Plasmas **6**, 2254 (1999).

Harris Instability in Intense One-Component Beams

- Detailed analytical and numerical studies* of the Harris instability have been carried out in the beam frame for electrostatic perturbations about the self-consistent Vlasov equilibrium

$$f_b^0(r,\mathbf{p}) = \frac{\widehat{n}_b}{(2\pi m_b T_{\perp b})} \exp\Big(-\frac{H_\perp}{T_{\perp b}}\Big) \frac{1}{(2\pi m_b T_{\| b})^{1/2}} \exp\Big(-\frac{p_z^2}{2m_b T_{\| b}}\Big)$$

  Here, $H_\perp = p_\perp^2/2m_b + (1/2)m_b w_f^3(x^2+y^2) + e_b\phi^0(r)$ is the single-particle Hamiltonian for the transverse particle motion, and $w_f = const.$ is the transverse focusing frequency.

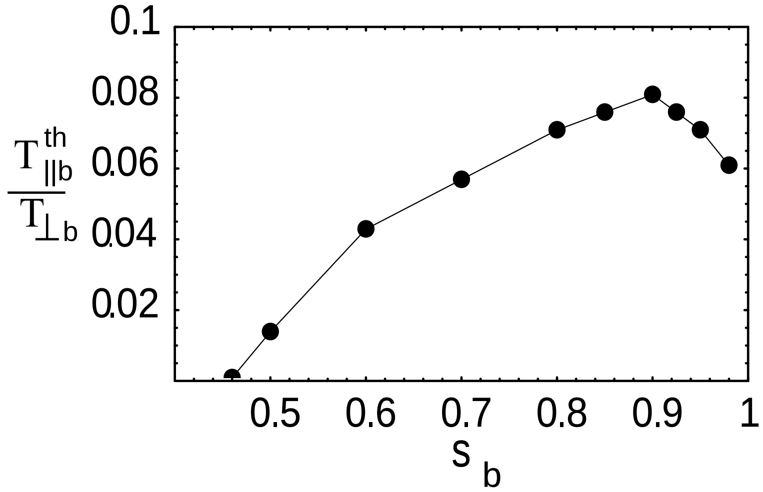
- The simulation studies make use of the 3D nonlinear delta-f code BEST. Simulations have been carried out for a wide range of energy anisotropy $T_{\| b}/T_{\perp b}$ and normalized beam intensity

$$s_b = \frac{\widehat{\omega}_{pb}^2}{2\omega_f^2}$$

  Here, $\widehat{\omega}_{pb} = (4\pi \widehat{n}_b e_b^2/\gamma_b m_b)^{1/2}$ is the ion plasma frequency.

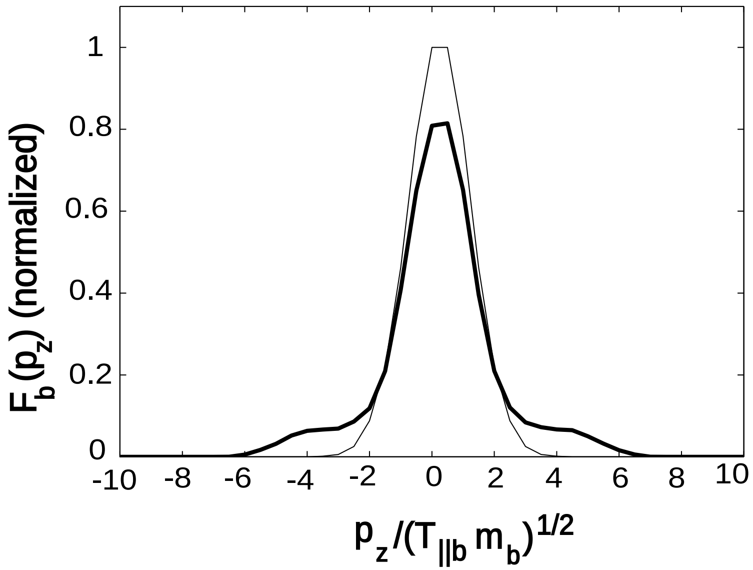
*E. A. Startsev, R. C. Davidson and H. Qin, Physics of Plasmas **9**, 3138 (2002); Laser and Particle Beams **20**, 585 (2002); Physical Review Special Topics on Accelerators and Beams **6**, 084401 (2003).

Harris Instability in Intense One-Component Beams



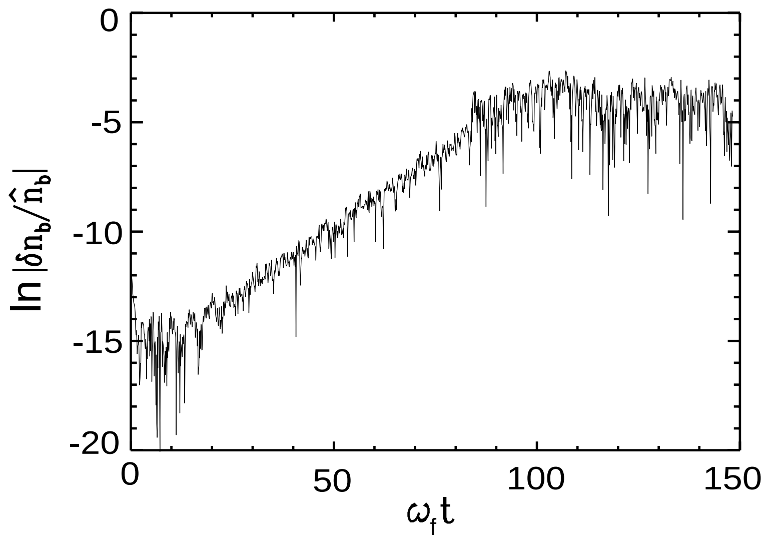
Longitudinal threshold temperature $T^{th}_{\|b}$ normalized to the transverse temperature $T_{\perp b}$ for the onset of instability plotted versus normalized beam intensity $s_b = \widehat{\omega}^2_{pb}/2\omega^2_f$.

Harris Instability in Intense One-Component Beams



Plot of average longitudinal momentum distribution $F_b(p_z, t)$ at time $t = 0$ (thin line) and $t = 150\omega_f^{-1}$ (thick line), for normalized beam intensity $s_b = 0.8$ and $T_{\|b}/T_{\perp b} = 0.02$.

Harris Instability in Intense One-Component Beams



Time history of the normalized density perturbation $\delta n_{\mathrm{max}}/\widehat{n}_b$ for $s_b = 0.8$ and $T_{\|b}/T_b = 0.02$ at fixed axial position z and radius $r = 0.3r_b$.

General Features of Harris Instability

- The anisotropy has to be sufficiently large, and the beam intensity $s_b = \widehat{\omega}_{pb}^2/2\omega_f^2$ sufficiently large for instability to occur.

- In the simulations, the growth rates and mode structure are in very good agreement with the analytical estimates during the linear stage of instability.

- The instability stabilizes nonlinearly through resonant wave-particle interactions and tail formation in the distribution function.

- A large remnant energy anisotropy remains after the instability saturates.

Weibel Instability in Intense One-Component Beams

- The electromagnetic Weibel instability* can play an important role in multispecies anisotropic plasmas and beam-plasma systems.

- However, the Weibel instability is not likely to play an important role in one-component nonneutral beams:
  - Constraints imposed by finite transverse geometry.
  - Electrostatic Harris instability has a much larger growth rate in the region where the Weibel instability exists.

*E. S. Weibel, Phys. Rev. Lett. **2**, 83 (1959).

Weibel Instability in Intense One-Component Beams

Assume axisymmetric perturbations ($\partial/\partial\theta = 0$) and transverse electromagnetic fields with polarization

$$\delta \mathbf{E}^T = \delta E_\theta \hat{\mathbf{e}}_\theta \ , \quad \delta \mathbf{B}^T = \delta B_r \hat{\mathbf{e}}_r + \delta B_z \hat{\mathbf{e}}_z$$

Analysis of the linearized Vlasov-Maxwell equations leads to an infinite dimension matrix dispersion equation*

$$\det\{D_{n,m}(\omega)\} = 0$$

which is valid for arbitrary normalized beam intensity

$$s_b = \frac{\widehat{\omega}_{pb}^2}{2\gamma_b^2 \omega_f^2}$$

and temperature anisotropy $T_{\parallel b}/T_{\perp b}$. Here, $\widehat{\omega}_{pb} = (4\pi \widehat{n}_b e_b^2/\gamma_b m_b)^{1/2}$ is the ion plasma frequency.

*E. A. Startsev and R. C. Davidson, Phys. Plasmas **10**, 4829 (2003).

Weibel Instability in Intense One-Component Beams

In the limit $T_{\parallel b}/T_{\perp b} \to 0$, the maximum growth rate of the Weibel instability is

$$(Im\omega)_{\text{max}} = 0.85\widehat{\omega}_{pb}\frac{v_{\perp b}^{th}}{c}$$

for $k_z^2 r_b^2 \gg 1$.

Detailed numerical analysis of matrix dispersion relation equation for finite $T_{\parallel b}$ shows that the Weibel instability is completely stabilized whenever $T_{\parallel b}$ exceeds the small threshold value $T_{\parallel b}^{th}$ defined by

$$\frac{T_{\parallel b}^{th}}{T_{\perp b}} = 0.2\frac{r_b^2 \widehat{\omega}_{pb}^2}{c^2} \ll 1$$

Electron-Ion Two-Stream (Electron Cloud) Instability

- A background population of electrons can result when energetic beam ions strike the chamber wall or ionize background gas atoms.

- Relative streaming motion of beam ions through the background electrons provides the free energy to drive the classical two-stream instability, appropriately modified to include the effects of dc space charge, relativistic kinematics, transverse beam geometry, etc.

- Experimental evidence for two-stream instability in proton machines such as the Proton Storage Ring (PSR) experiment at Los Alamos National Laboratory.

Electron-Ion Two-Stream (Electron Cloud) Instability

- Instability is three-dimensional in nature, with strongest growth exhibited by dipole-mode perturbations with azimuthal mode number $m = 1$. This is a common feature of experiments, analytical theory, and numerical simulations.

- 3D nonlinear delta-f simulations by Qin, et al., using the BEST code, have investigated detailed properties of this instability for a wide range of system parameters. Simulations use a smooth focusing model in which the electrons are confined in the transverse plane by the (excess) space charge of the beam ions.

- Detailed analytical investigations of linear stability properties have also been carried out for arbitrary multipole perturbations about a Kapchinskij-Vladimirskij (KV) distribution with step-function density profile.

Electron-Ion Two-Stream (Electron Cloud) Instability

As a simple example, consider dipole-mode perturbations about a KV distribution. Assume cold beam ions with axial velocity V_b propagate through a stationary electron background ($V_e = 0$).

Dispersion Relation

$$[(\omega - k_z V_b)^2 - \omega_b^2][\omega^2 - \omega_e^2] = \omega_c^4 \ ,$$

Definitions

$$\begin{aligned}
\omega_c^4 &= \frac{1}{4} f \left(1 - \frac{r_b^2}{r_w^2}\right)^2 \frac{\gamma_b m_b}{Z_b m_e} \widehat{\omega}_{pb}^4 \\
\omega_b^2 &= \omega_f^2 + \frac{1}{2}\widehat{\omega}_{pb}^2 \left(f - \frac{1}{\gamma_b^2}\frac{r_b^2}{r_w^2}\right) \\
\omega_e^2 &= \frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\widehat{\omega}_{pb}^2 \left(1 - f\frac{r_b^2}{r_w^2}\right)
\end{aligned}$$

where

$$f = \text{fractional charge neutralization}$$

$$\widehat{\omega}_{pb} = \left(\frac{4\pi \widehat{n}_b Z_b^2 e_b^2}{\gamma_b m_b}\right)^{1/2} = \text{ion plasma frequency}$$

Electron-Ion Two-Stream (Electron Cloud) Instability

BEST simulations have been carried out for 3D perturbations about distribution functions ($j = b, e$)

$$f_j^0(r, \mathbf{p}) = \frac{\widehat{n}_j}{2\pi\gamma_j m_j T_{\perp j}} \exp\Big(-\frac{H_{\perp j}}{T_{\perp j}}\Big) G_j(p_z)$$

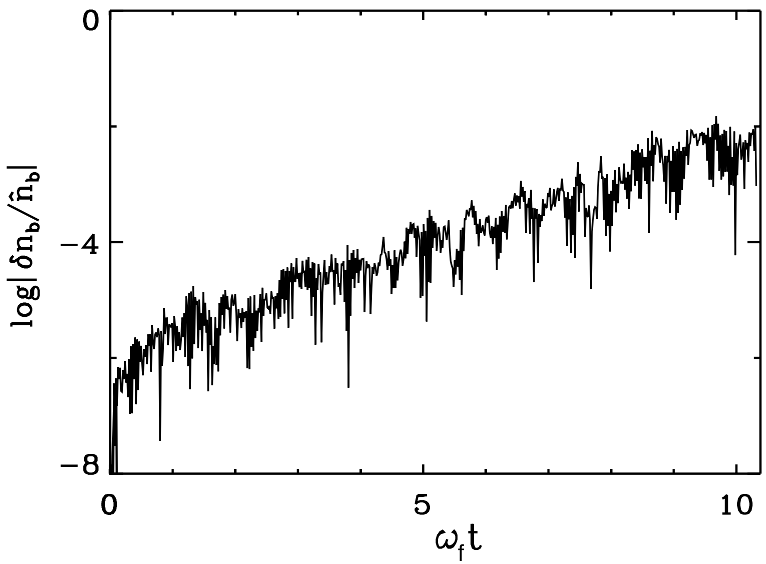
For the beam ions, take $G_b(p_z)$ to be a drifting Maxwellian centered at $p_z = \gamma_b m_b V_b$. For the background electrons take $G_e(p_z)$ to be a Maxwellian centered at $p_z = 0$.

Illustrative Parameters

Linearized δF simulations for 2.5 GeV cesium ion beam with

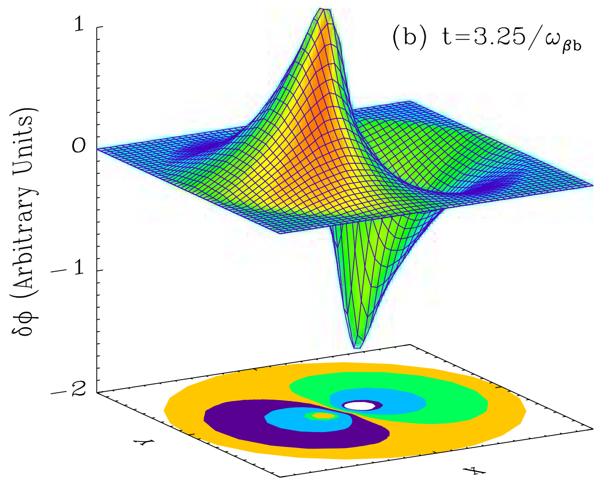
$$f = \frac{\widehat{n}_e}{\widehat{n}_b} = 0.1 \ , \quad \frac{T_{b\perp}}{\gamma_b m_b V_b^2} = 1.1 \times 10^{-6} \ , \quad \frac{T_{e\perp}}{\gamma_b m_b V_b^2} = 2.47 \times 10^{-6}$$

Electron-Ion Two-Stream (Electron Cloud) Instability



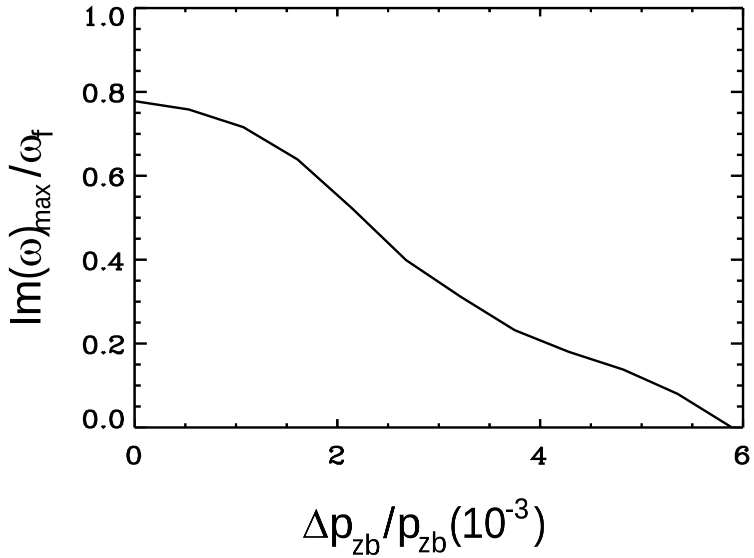
Time history of perturbed density $\delta n_b/\tilde{n}_b$ at a fixed spatial location. After an initial transition period, the $m = 1$ dipole-mode perturbation grows exponentially.

Electron-Ion Two-Stream (Electron Cloud) Instability



The $x-y$ projection (at fixed value of z) of the perturbed electrostatic potential $\delta\phi(x, y, t)$ for the ion-electron two-stream instability growing from a small initial perturbation, shown at $\omega_f t = 3.25$.

Electron-Ion Two-Stream (Electron Cloud) Instability



The maximum linear growth rate $(Im\omega)_{\text{max}}$ of the ion-electron two-stream instability decreases as the longitudinal momentum spread of the beam ions increases.

Growth Rate Reduction Mechanisms

- Axial momentum spread in the beam ions.
- Proximity of a conducting wall.
- Reduction in fractional charge neutralization.
- Rounded beam density profiles
  - Spread in transverse oscillation frequency.

Candidate Instabilities

- Resistive Hose
- Multispecies Weibel
- Multispecies Two-Stream

Resistive Hose Instability

Assumptions

- Step-function density profile.
- Transverse electromagnetic dipole-mode perturbations

$$\delta \mathbf{E}^T = \delta E_z \widehat{\mathbf{e}}_z \ , \quad \delta \mathbf{B}^T = \delta B_r \widehat{\mathbf{e}}_r + \delta B_\theta \widehat{\mathbf{e}}_\theta$$

- Charge neutralizing plasma background

$$\sum_{j=b,e,i} n_j^0(r) e_j = 0$$

- Partial current neutralization

$$J_{zp} = -f_m(\widehat{n}_b e_b \beta_b c)$$

Resistive Hose Instability

Dipole-mode perturbations with

$$\delta E_z(\mathbf{x}, t) = \delta \widehat{E}_z(r) \exp[i(\theta - \Omega z/V_b - \omega\tau)]$$

where $\Omega = \omega - k_z V_b$.

Dispersion relation is given by

$$\frac{\widehat{\omega}_{pb}^2 \beta_b^2}{\Omega^2 - \omega_\beta^2} = -\kappa_p r_b \frac{J_1'(\kappa_p r_b)}{J_1(\kappa_p r_b)} - \frac{r_w^2 + r_b^2}{r_w^2 - r_b^2} \ ,$$

where

$$\omega_\beta^2 = \frac{1}{2} \widehat{\omega}_{pb}^2 \beta_b^2 (1 - f_m) \ ,$$

$$\kappa_p^2(\omega) r_b^2 = \frac{8 i \omega \tau_d}{(1 - i\omega/\nu_c)} \ .$$

Illustrative Example

- 1 kA, 2.5 GeV cesium beam.
- $\hat{n}_b = 3.4 \times 10^{11} cm^{-3}$; beam radius $r_b = 1cm$.
- Zero plasma return current ($f_m = 0$).
- For $T_e = 1$ eV and $\hat{n}_e = 10^{12} cm^{-3}$ we obtain:

$$\sigma_p = 3 \times 10^{12} s^{-1} \text{(conductivity)}$$

$$\tau_d = \frac{\pi \sigma_p r_b^2}{2c^2} = 5 \times 10^{-9} s \text{(magnetic decay time)}$$

- Resulting growth rate is

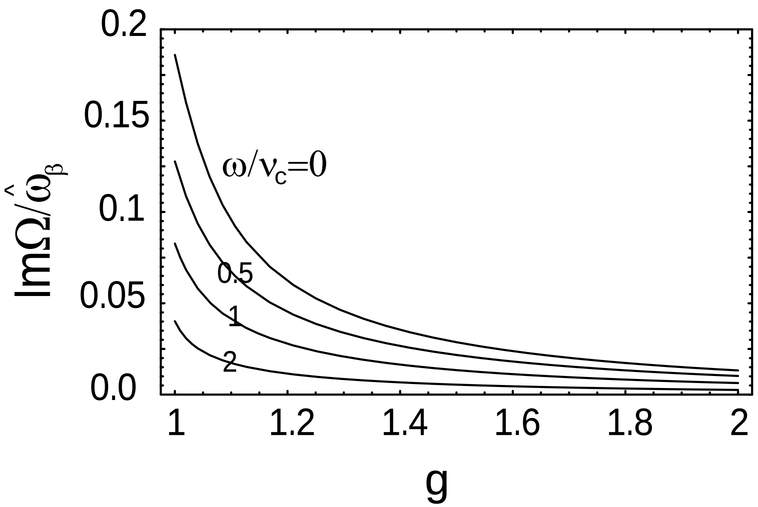
$$Im\Omega = 0.13\omega_\beta$$

where $\omega_\beta = 9.2 \times 10^6 s^{-1}$.

Growth Rate Reduction Mechanisms

- Proximity of a conducting wall.
  - Increasing $g = \left(1 - \frac{r_b^2}{r_w^2}\right)^{-1}$.
- Increasing the value of $|\omega|/\nu_c$.
- Decreasing the value of fractional current neutralization f_m.
  - Stabilizing influence of B_θ.
- Rounded beam density profile.
  - Reduces number of resonant beam particles.
- Increase electron temperature.
  - Higher plasma conductivity.

Resistive Hose Instability



Instability growth rate is reduced by increasing values of $|\omega|/\nu_c$ and by the proximity of a conducting wall, i.e., increasing $g = \left(1 - \frac{r_b^2}{r_w^2}\right)^{-1}$.

Multispecies Weibel Instability

In the collisionless regime, the large directed kinetic energy of the beam ions propagating through a background plasma provides the free energy to drive the electromagnetic Weibel instability.

Assumptions

- Charge and current neutralization

$$\sum_{j=b,e,i} n_j^0(r)e_j = 0 \quad \text{and} \quad \sum_{j=b,e,i} n_j^0(r)e_j\beta_j c = 0$$

- Electromagnetic perturbations with $\partial/\partial\theta = 0 = \partial/\partial z$ and polarization

$$\delta\mathbf{E} = \delta E_r \hat{\mathbf{e}}_r + \delta E_z \hat{\mathbf{e}}_z \ , \quad \delta\mathbf{B} = \delta B_\theta \hat{\mathbf{e}}_\theta$$

Multispecies Weibel Instability

Express

$$\delta E_z(r,t) = \delta \widehat{E}_z(r)\, \mathrm{exp}(-i\omega t)$$

Obtain the eigenvalue equation

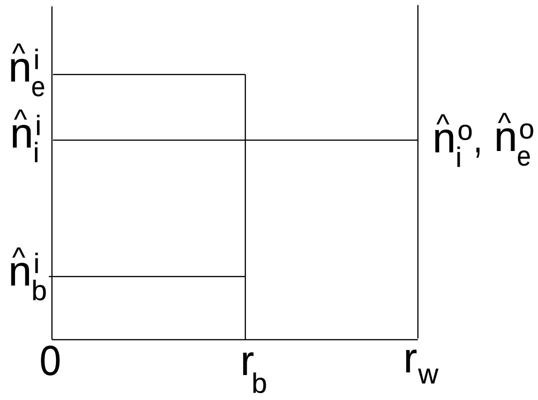
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(1 + \sum_{j=b,e,i} \frac{\beta_j^2 \omega_{pj}^2(r)}{\omega^2} + \frac{(\sum_{j=b,e,i} \beta_j \omega_{pj}^2(r))^2}{\omega^2[\omega^2 - \sum_{j=b,e,i} \omega_{pj}^2(r)]} \right) \frac{\partial}{\partial r} \delta \widehat{E}_z \right] + \left(\frac{\omega^2}{c^2} - \sum_{j=b,e,i} \frac{\omega_{pj}^2(r)}{\gamma_j^2 c^2} \right) \delta \widehat{E}_z = 0$$

where $\omega_{pj}(r) = [4\pi n_j^0(r) e_j^2/\gamma_j m_j]^{1/2}$ and $\gamma_j = (1-\beta_j^2)^{-1/2}$.

Slow-wave Weibel instability driven by the terms proportional to

$$\sum_{j=b,e,i} \beta_j^2 \omega_{pj}^2(r) \quad \text{and} \quad \sum_{j=b,e,i} \beta_j \omega_{pj}^2(r)$$

Multispecies Weibel Instability



An intense ion beam with step-function density profile propagates through background plasma with uniform density.

Treating the plasma ions as stationary ($\beta_i = 0$) and assuming local charge and current neutralization gives

$$\widehat{n}_i^i = Z_b \widehat{n}_b^i + Z_i \widehat{n}_i^i$$

$$\beta_e = \frac{Z_b \widehat{n}_b^i}{Z_b \widehat{n}_b^i + Z_i \widehat{n}_i^i} \beta_b$$

Multispecies Weibel Instability

For the case of uniform density profiles the eigenvalue equation can be solved exactly to obtain a closed dispersion relation for the complex oscillation frequency ω.

For perturbations with short transverse wavelength the characteristic growth rate of the Weibel instability scales as $Im\omega \sim \Gamma_w$ where

$$\Gamma_w^2 \equiv \beta_e^2 \widehat{\omega}_{pi}^{i^2} + (\beta_b - \beta_e)^2 \widehat{\omega}_{pb}^{i^2}$$

and $\widehat{\omega}_{pj}^i = (4\pi \widehat{n}_j^i e_j^2/\gamma_j m_j)^{1/2}$.

Multispecies Weibel Instability

The full dispersion relation has been solved numerically over a wide range of beam-plasma parameters.

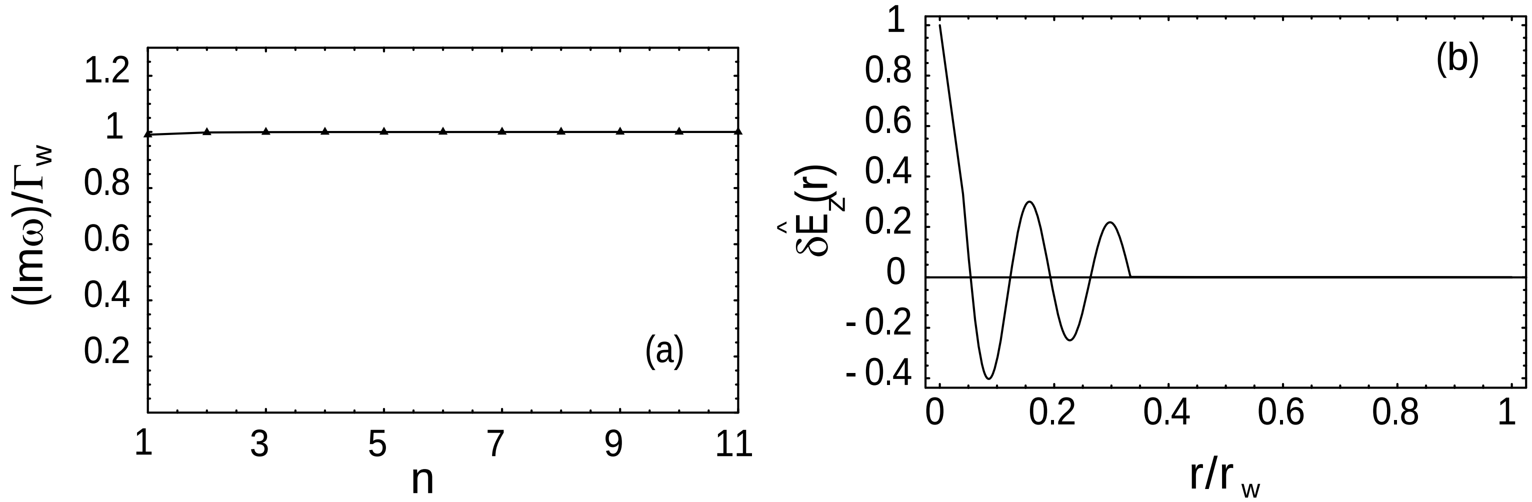
As an illustrative example consider an intense cesium ion beam ($Z_b = 1$) propagating through background argon plasma ($Z_i = 1$) with

$$\beta_b = 0.2 \ , \quad \beta_i = 0 \ , \quad \beta_e = 0.1$$

$$\hat{n}_b^i = \hat{n}_e^i/2 = \hat{n}_i^i$$

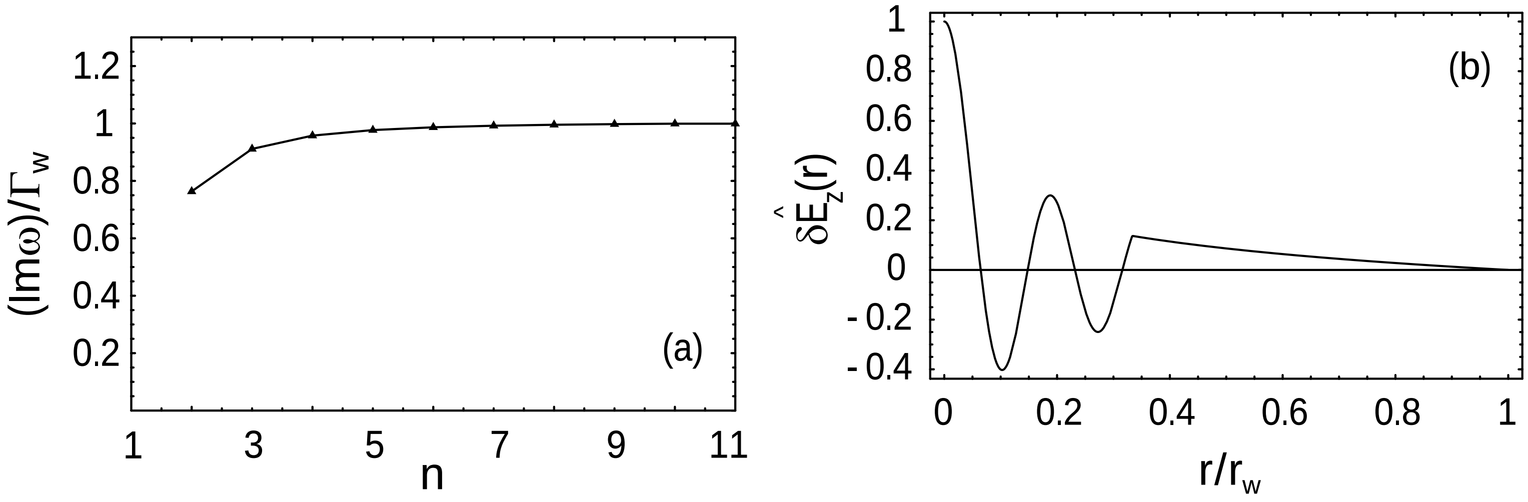
The background plasma provides complete charge and current neutralization.

Multispecies Weibel Instability



Plots of (a) Weibel instability growth rate $(Im\omega)/\Gamma_w$ versus mode number n, and (b) eigenfunction $\delta\hat{E}_z(r)$ versus r/r_w for $n = 5$. System parameters are $r_b = r_w/3$, $\hat{\omega}_{pe}^i r_b/c = 1/3$ and $\hat{n}_j^0 = 0$ (vacuum region outside beam).

Multispecies Weibel Instability



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Multispecies Weibel Instability

- An effective means to reduce the growth rate of the Weibel instability is to decrease the value of fractional current neutralization f_m.
- The B_θ self-magnetic field constrains the transverse particle motion and likelihood of filamentation in the transverse plane.

Multispecies Two-Stream Instability

In the collisionless regime, the large directed kinetic energy of the beam ions relative to the background plasma components also provides the free energy to drive the electrostatic two-stream instability.

Assumptions

- Charge and current neutralization

$$\sum_{j=b,e,i} n_j^0(n)e_j = 0 \quad \text{and} \quad \sum_{j=b,e,i} n_j^0(r)e_j\beta_j c = 0$$

- Longitudinal electrostatic perturbations with $\delta \mathbf{E}^L = -\nabla \delta\phi$ and $\partial/\partial\theta = 0$

$$\delta \mathbf{E}^L = \delta E_r \hat{\mathbf{e}}_r + \delta E_z \hat{\mathbf{e}}_z$$

Multispecies Two-Stream Instability

Express

$$\delta\phi(r,z,t) = \widehat{\delta\phi}(r)\,\mathrm{exp}(ik_z z - i\omega t)$$

Obtain eigenvalue equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(1-\sum_{j=b,e,i}\frac{\omega_{pj}^2(r)/\gamma_j^2}{(\omega-k_zV_{zj})^2}\right)\frac{\partial}{\partial r}\widehat{\delta\phi}\right] - k_z^2\left(1-\sum_{j=b,e,i}\frac{\omega_{pj}^2(r)/\gamma_j^2}{(\omega-k_zV_{zj})^2}\right)\widehat{\delta\phi} = 0$$

Here, $\omega_{pj}(r) = [4\pi n_j^0(r)e_j^2/\gamma_j m_j]^{1/2}$ is the relativistic plasma frequency, $V_{zj} = \beta_j c = const.$ is the average axial velocity of component j ($j = b, e, i$), and $\gamma_j = (1-\beta_j^2)^{-1/2}$ is the relativistic mass factor.

Two-stream instability is driven by the relative axial motion V_{zj} of the beam-plasma components.

Multispecies Two-Stream Instability

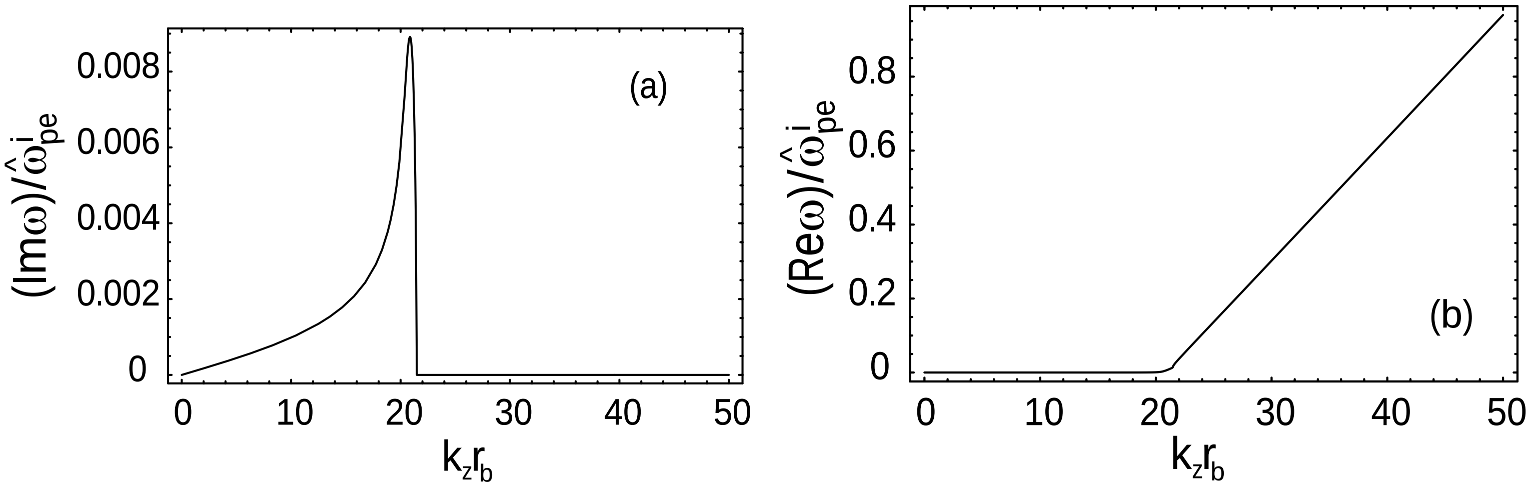
Obtain electrostatic dispersion relation

$$D(k_z,\omega) = 1 - g_0 \sum_{j=b,e,i} \frac{\widehat{\omega}_{pj}^{i2}/\gamma_j^2}{(\omega - k_z V_{zj})^2} - (1-g_0) \sum_{j=e,i} \frac{\widehat{\omega}_{pj}^{02}}{\omega^2} = 0$$

where g_0 is the geometric factor defined by

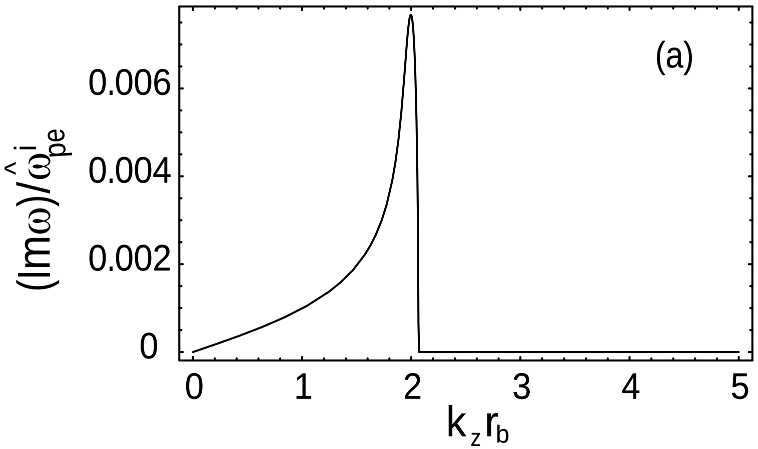
$$g_0 = k_z r_b I_0'(k_z r_b) I_0(k_z r_b) \left[\frac{K_0(k_z r_b)}{I_0(k_z r_b)} - \frac{K_0(k_z r_w)}{I_0(k_z r_w)} \right] \ .$$

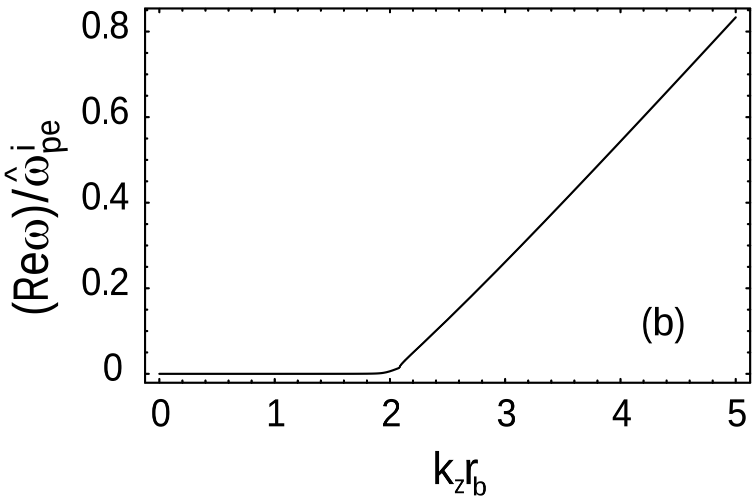
Multispecies Two-Stream Instability



Plots of (a) $(Im\omega)/\widehat{\omega}^i_{pe}$ and (b) $(Re\omega)/\widehat{\omega}^i_{pe}$ versus $k_z r_b$ calculated from the two-stream dispersion relation for $r_b = r_w/3$, $\beta_b = 0.2$, $\beta_e = 0.1$ and $\widehat{\omega}^2_{pe} r_b/c = 3$ in the absence of plasma outside the beam-plasma channel.

Multispecies Two-Stream Instability





Plots of (a) $(Im\omega)/\widehat{\omega}^i_{pe}$ and (b) $(Re\omega)/\widehat{\omega}^i_{pe}$ versus $k_z r_b$ calculated from the two-stream dispersion relation for $r_b = r_w/3$, $\beta_b = 0.2$, $\beta_e = 0.1$ and $\widehat{\omega}^2_{pe} r_b/c = 1/3$ in the absence of plasma outside the beam-plasma channel.

Multispecies Two-Stream Instability

- A small axial momentum spread of the beam ions and the plasma ions leads to a reduction in the growth rate of the two-stream instability.
- Two-stream is likely to lead to a longitudinal heating of the plasma electrons in the nonlinear regime.

Conclusions

- A wide variety of collective instabilities in intense beams and beam-plasma systems have been investigated.
- Growth rate reduction (or elimination) mechanisms have been identified.
- Numerical simulations are playing a critical role in determining threshold conditions and nonlinear dynamics.
- See related publications at http://nonneutral.pppl.gov.
- Related papers at this Symposium by Kaganovich, Qin, Lee and Startsev.