Perturbative Particle Simulation Studies of Periodically Focused Intense Charged Particle Beams. 

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Abstract

High intensity charged particle beam propagation in a periodic focusing lattice has been studied numerically using a model in which the beam equilibrium and dynamic behavior are described self-consistently by the nonlinear Vlasov-Maxwell equations. For a long coasting which is inhomogeneous in the transverse direction, the solutions to the Vlasov-Maxwell equations for periodic focusing configurations can only be determined numerically. To carry out this investigation, the Beam Equilibrium Stability and Transport (BEST) code which uses a 3D low-noise perturbative particle simulation method, has been extended. The scheme begins with a smooth-focusing lattice which is the smooth-focusing approximation for the periodic lattice, and adiabatically replaces the smooth-focusing lattice by the periodic lattice. With this approach, periodic solenoidal configurations have been investigated using a slow turn-on time to minimize beam mismatch, and periodic quadrupole configurations are now being studied.
Objective and Method

- The objective is to find practical solutions for Intense Charged Particle Beam (ICPB) in Periodic Focusing Configurations so that we can determine the detailed equilibrium, stability, and transport properties;

- BEST code is used to numerically solve the Vlasov-Maxwell equations;

- With the $\delta f$ simulation method, the BEST code succeeds in reducing the noise by a factor of $\frac{f}{\delta f}$;

- To apply the $\delta f$ simulation method, the numerical scheme begins with a smooth-focusing lattice model for the periodic lattice, and adiabatically replaces the smooth-focusing lattice by the periodic lattice.
Analytical approach:

- The **nonlinearity** of the Vlasov-Maxwell equations makes it impossible to find a general analytical solution;

- The only analytical solution is the Kapchinskij-Vladimirskij distribution, which is of limited practical interest;

- Using **Hamiltonian Averaging Techniques**, it is possible to obtain approximate analytical solutions.

- Numerically, the $\delta f$ simulation method is promising. To apply this method, a "quasi-equilibrium" solution is needed.
The Vlasov-Maxwell equations in a two-dimensional slice model are:

\[
\left\{ \frac{\partial}{\partial s} + x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} - \left( \kappa_x(s) x + \frac{\partial \psi}{\partial x} \right) \frac{\partial}{\partial x'} \right. \\
- \left. \left( \kappa_y(s) y + \frac{\partial \psi}{\partial y} \right) \frac{\partial}{\partial y'} \right\} F_b = 0,
\]

and

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = -\frac{2\pi K_b}{N_b} \int dx' dy' F_b,
\]
in which \(\kappa_x(s)\) and \(\kappa_y(s)\) are lattice functions, and the transverse focusing force on a beam particle is given by \(F_{foc} = -[\kappa_x(s) x\hat{e}_x + \kappa_y(s) y\hat{e}_y]\).

Here, \(K_b = 2N_b e_b^2 / \gamma_b^3 m_b \beta_b^2 c^2\) is the self-field perveance, and \(\psi = e_b \phi / \gamma_b^3 m_b \beta_b^2 c^2\) is the normalized self-field potential.
The magnetic field is given by:

\[ \mathbf{B}_{\text{sol}} (\mathbf{x}) = -\frac{1}{2} B_z' (z) (x \hat{e}_x + y \hat{e}_y) + B_z (z) \hat{e}_z \]

\[ \mathbf{A}_{\text{sol}} (\mathbf{x}) = \frac{1}{2} B_z (z) (x \hat{e}_y - y \hat{e}_x) = \frac{1}{2} B_z (z) r \hat{e}_\theta \]

Periodic variation:

\[ B_z (z + S) = B_z (z) \]

The lattice functions are given by:

\[ \kappa_x (s) = \kappa_y (s) = \kappa_z (s) = \left( \frac{e_b B_z (z)}{2 \gamma_b m_b \beta_b c^2} \right)^2 \]

In general,

\[ \int_{s_0}^{s_0 + S} ds \kappa_z (s) \neq 0 \]
The magnetic field is given by:

\[
B_q(x) = B'_q(z) (y\hat{e}_x + x\hat{e}_y),
\]

\[
A_q(x) = -\frac{1}{2}B'_q(z) (x^2 - y^2) \hat{e}_z.
\]

Periodic variation:

\[
B'_q(z + S) = B'_q(z)
\]

The lattice functions are given by:

\[
\kappa_x(s) = -\kappa_y(s) \equiv \kappa_q(s) = \frac{e_b B'_q(s)}{\gamma_b m_b \beta_b c^2}
\]

\[
\kappa_q(s) \text{ satisfies: }
\int_{s_0}^{s_0+S} ds \kappa_q(s) = 0
\]
Define $F^0_b$ as a zeroth-order equilibrium distribution function which satisfies
\[
\frac{\partial F^0_b}{\partial s} + \mathbf{X}' \cdot \frac{\partial F^0_b}{\partial \mathbf{X}} + \left(-\kappa_{sf} \mathbf{X} - \frac{\partial \psi^0}{\partial \mathbf{X}}\right) \cdot \frac{\partial F^0_b}{\partial \mathbf{X}'} = 0,
\]
\[
\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) \psi^0 = -\frac{2\pi K_b}{N_b} \int dX' dY' F^0_b,
\]
$k_{sf}$ is obtained in the smooth-focusing approximation.

For a solenoidal lattice,
\[
k_{sf} = \bar{\kappa}_z + \left[\left\langle \left(\int_{s_0}^s ds \delta \kappa_z(s) \right)^2 \right\rangle - \left\langle \left(\int_{s_0}^s ds \delta \kappa_z(s) \right) \right\rangle^2 \right]
\]
where $\bar{\kappa}_z \equiv (1/S) \int_{s_0}^{s_0+S} ds \kappa_z(s)$.

For a quadrupole lattice,
\[
k_{sf} = \left\langle \left(\int_{s_0}^s ds \kappa_q(s) \right)^2 \right\rangle - \left\langle \left(\int_{s_0}^s ds \kappa_q(s) \right) \right\rangle^2
\]
Simulation Scheme II

Define \( \delta F_b \) and \( w \) as \( \delta F_b \equiv F_b - F_b^0 \), \( w \equiv \frac{\delta F_b}{F_b} \). The dynamical equation for \( w \) is given by

\[
\frac{d}{ds} w = (1 - w) \left\{ \left( \delta \kappa_x (s) X + \frac{\partial}{\partial X} \delta \psi \right) \frac{1}{F_b^0} \frac{\partial F_b^0}{\partial X'} + \left( \delta \kappa_y (s) Y + \frac{\partial}{\partial Y} \delta \psi \right) \frac{1}{F_b^0} \frac{\partial F_b^0}{\partial Y'} \right\}
\]

with \( \delta \psi (X, Y, s) \) determined by

\[
\left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) \delta \psi = -\frac{2\pi K_b}{N_b} \int dX' dY' w F_b = -\frac{2\pi K_b}{N_b} \int dX' dY' \frac{w}{1 - w} F_b^0
\]

Here, at \( s = 0 \), \( \delta \kappa_x (s) = \delta \kappa_y (s) = 0 \);

For \( s \) large enough,

\( \delta \kappa_x (s) = \delta \kappa_y (s) \rightarrow \kappa_z (s) - \kappa_{sf} \)

for a solenoidal lattice, and

\( \delta \kappa_x (s) \rightarrow \kappa_q (s) - \kappa_{sf}, \delta \kappa_y (s) \rightarrow -\kappa_q (s) - \kappa_{sf} \)

for a quadrupole lattice.
Turning on the Solenoidal Lattice

\[ B_z(s) = B_{z0} \left[ 1 + \frac{\Delta_s}{2} \cos \left( \frac{2\pi s}{S} \right) \right], \]

where \( \Delta_s = \Delta_m \left[ 1 - \exp \left( -\frac{s}{NS} \right) \right] \), and N is the number of lattice periods for turn-on. In the figure, N=10.
Figure: Time evolution of root mean square radius for the adiabatic model I

\[ s_b = 0.2, \Delta_m = 0.2, N = 10 \]

where \( s_b = \frac{\omega_{pb}^2}{2\gamma_b^2 \omega_{\beta\perp}^2} \) is the normalized beam intensity.
Figure: Time evolution of root mean square radius for the adiabatic model I
Figure: FFT of $\delta r^2 \left( s \right)/r_{b0}^2$ for $N = 10$. The left figure is FFT from $s = 0$ to $s = 30S$, and the right figure is FFT from $s = 470S$ to $s = 500S$.

After sufficient time, we obtain a well-matched periodically-focused beam.
Figure: Time evolution of root mean square radius for the adiabatic model II

\[ s_b = 0.2, \Delta_m = 0.6, N = 10 \]
Figure: Time evolution of root mean square radius for the adiabatic model II for $\Delta m = 0.6$
Figure: FFT of $\delta r^2 (s) / r_{b0}^2$ for $N = 10$ and $\Delta_m = 0.6$. The left figure is FFT from $s = 0$ to $s = 30S$, and the right figure is FFT from $s = 470S$ to $s = 500S$.

After sufficient time, we also obtain a well-matched periodically-focused beam.
Figure: Time evolution of root mean square radius for the adiabatic model III

\[ s_b = 0.2, \Delta_m = 1.8, N = 10 \]
Figure: Time evolution of root mean square radius for the adiabatic model III for $\Delta_m = 1.8$
Figure: Time evolution of root mean square radius for $N = 10$ (slow turn-on)

\[ s_b = 0.9, \Delta_m = 0.2 \]
Figure: Time evolution of root mean square radius for $N = 0.1$ (fast turn-on)

$s_b = 0.9, \Delta_m = 0.2$
Figure: FFT of $\delta r^2 (s) / r_{b0}^2$ from $s = 0$ to $s = 30S$. The left figure is for $N = 10$ (slow turn-on), and the right figure is for $N = 0.1$ (fast turn-on).
**FFT of the Beam Radius Perturbation for Different $N$**

Figure: FFT of $\delta r^2 (s) / r_{b0}^2$ from $s = 0$ to $s = 500S$. The left figure is for $N = 10$ (slow turn-on), and the right figure is for $N = 0.1$ (fast turn-on).

Larger $N \Rightarrow$ better beam matching.
Figure: FFT of $\delta r^2 (s)/r_{b0}^2$ from $s = 0$ to $s = 30S$. The left figure is for $s_b = 0.2$ and the right figure is for $s_b = 0.9$. Both are for $N = 10$ (slow turn-on).

Larger $s_b \Rightarrow$ somewhat better beam matching.
FFT of the Beam Radius Perturbation for Different $s_b$

Figure: FFT of $\delta r^2(s)/r_{b0}^2$ from $s = 0$ to $s = 500S$. The left figure is for $s_b = 0.2$ and the right figure is for $s_b = 0.9$. Both are for $N = 10$ (slow turn-on).

Larger $s_b \Rightarrow$ somewhat better beam matching.
Turning on the Quadrupole Lattice

\[ B_q(s) = \frac{B_q(s)}{B_{q0}} \]

\[ \delta \kappa_q / \kappa_{q0} \]

\[ B_q(s) = B_{q0} \Delta_s \sin \left( \frac{2\pi s}{S} \right) \]

where \( \Delta_s = \left[ 1 - \exp \left( -\frac{s}{NS} \right) \right] \), and N is the number of lattice periods for turn-on. In the figure, N=10.
Figure: Time evolution in the \( x \) direction for \( N = 20 \) (slow turn-on). In the figure, \( x_{\text{rms}} \) is the root mean square \( x \) for all the particles.

\[
s_b = 0.1, \quad \sigma_v = 57.3^\circ
\]

where \( \sigma_v \) is the vacuum phase advance.
Adiabatic Model for the Quadrupole Lattice

Figure: Time evolution in the \( x \) direction for \( N = 20 \) (slow turn-on)
Figure: FFT of $\sqrt{2}x_{rms}(s)/r_{b0}$ for $N = 20$. The left figure is FFT from $s = 0$ to $s = 30S$, and the right figure is FFT from $s = 270S$ to $s = 300S$. 
Figure: Time evolution in the $x$ direction for $N = 20$ (slow turn-on). In the figure, $x_{rms}$ is the root mean square $x$ for all the particles.

For beam with $s_b = 0.1$, the $\delta f$ simulation of the Vlasov-Maxwell equation agrees quite well with the simulation of the envelope equation for the same adiabatic turning-on model.
Figure: Time evolution in the $x$ direction for $N = 20$ (slow turn-on)
Conclusion and Future Plan

For the solenoidal lattice configuration:

- By adiabatically replacing the smooth-focusing lattice with the periodic solenoidal lattice in the $\delta f$ simulation, we can obtain good matched beam after sufficient time.
- With the slower turn-on model, we can obtain better beam matching than the faster turn-on model.
- The beam with higher intensity matches the solenoidal lattice better than the beam with lower intensity.

For the quadrupole lattice configuration:

- When the normalized intensity $s_b = 0.1$, $\delta f$ simulation of the Vlasov-Maxwell equation agrees with the simulation of the envelope equations. Similar to the solenoidal case, we can obtain rather good matched beam after sufficient time. Actually when $s_b$ is small enough ($s_b < 0.7$), we can always obtain good matched beam after sufficient time.

- More simulations for the quadrupole lattice configuration will be carried out to investigate the dynamics of the quadrupole focused beam.