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Thermodynamic bounds on nonlinear electrostatic perturbations in intense charged particle beams

Nikolas C. Logan and Ronald C. Davidson
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543, USA

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This paper places a lowest upper bound on the field energy in electrostatic perturbations in single-species charged particle beams with initial temperature anisotropy $(T_{\parallel}/T_{\perp} < 1)$. The result applies to all electrostatic perturbations driven by the natural anisotropies that develop in accelerated particle beams, including Harris-type electrostatic instabilities, known to limit the luminosity and minimum spot size attainable in experiments. The thermodynamic bound on the field perturbation energy of the instabilities is obtained from the nonlinear Vlasov-Poisson equations for an arbitrary initial distribution function, including the effects of intense self-fields, finite geometry, and nonlinear processes. This paper also includes analytical estimates of the nonlinear bounds for space-charge-dominated and emittance-dominated anisotropic bi-Maxwellian distributions. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4737180]

I. INTRODUCTION

The physics of high-density, high-current non-neutral particle beams is characterized by intense nonlinear self-fields that often make analytical descriptions difficult. These charged particle beams are essential components of high energy accelerators and associated transport and storage systems.[1–5] They have become crucial scientific tools for research on spallation neutron sources, nuclear physics, beam-driven high energy density physics, and heavy ion inertial fusion drivers. Thus, much effort has been devoted to understanding the underlying physics of the nonlinear processes occurring in these beams. An important, tractable approach to solving the detailed dynamics of such systems is often to rely on advanced numerical tools such as particle-in-cell (PIC) simulations,[6–8] eigenmode codes,[9,10] and Monte Carlo codes[11–14] which can simulate the linear and nonlinear phases of instabilities that may cause a degradation of beam quality.

One example of the instabilities that occur in these systems is the electrostatic Harris instability,[9–12,15,16] which is driven by the strong temperature anisotropy ($T_{\parallel}/T_{\perp} \ll 1$, where the subscripts $\parallel$ and $\perp$ denote parallel and perpendicular to the beam propagation) that develops naturally in the frame of an accelerated charged particle beam. Much numerical work has been carried out to characterize this type of instability,[9–12,17–19] and it is therefore of particular importance to develop an analytical framework for comparison with these results. Previous theoretical work has concentrated on investigations of the detailed dynamics of beams with Kapchinskij-Vladimirskij (KV) distributions,[20,21] and two-temperature bi-Maxwellian distributions.[11,12] The formalism presented here is an important generalization of the Fowler bound[22,23] that fully incorporates effects of the strong self-fields, finite geometry, and nonlinear behavior of intense non-neutral particle beams.

In this paper, we consider a long, coasting single-species charged particle beam surrounded by a perfectly conducting, cylindrical wall. The beam properties are assumed to be periodic in the axial direction, and the theoretical model is based on the nonlinear Vlasov-Poisson equations. The present generalization of Fowler's method[22,23] makes use of the conservation of energy per unit length U, entropy per unit length S, and line density N. That is, the Helmholtz free energy F is shown to be a conserved quantity under these conditions. A field perturbation energy is then defined $(\epsilon_F \equiv \int d^3x(\nabla\delta\phi)^2/8\pi)$ in reference to the state corresponding to the minimum Helmholtz free energy, and any change in the field perturbation energy is bounded from above using this conservation constraint. The minimization of this upper bound, with respect to the distribution function, gives a *lowest upper bound* on the field perturbation energy. The results obtained here are applicable to any specified initial distribution, including anisotropic beams characterized by the Harris instability, and illustrative applications assuming initial bi-Maxwellian distributions are presented.

The organization of this paper is as follows. The theoretical model and definitions are provided in Sec. II, followed by a review of the unique properties of the thermal equilibrium reference distribution in Sec. III. The nonlinear bound on electrostatic perturbations in charge particle beams is calculated in Sec. IV. Finally, Sec. V demonstrates the utility of this result in simple analytical limits, and the conclusions are summarized in Sec. VI.

II. THEORETICAL MODEL AND THE HELMHOLTZ FREE ENERGY

The analysis in this paper places a limit on the severity of electrostatic instabilities that occur in intense anisotropic charged particle beams. The work assumes a single-species charged particle beam of particles with mass m_b and charge e_b. The long, coasting beam is confined transversely by an applied focusing force $\mathbf{F}_f = -\nabla\psi_f(\mathbf{x})$, where $\psi_f(\mathbf{x})$ is the effective smooth-focusing potential at position $\mathbf{x}$. The analysis is carried out in the beam frame, where the velocity of particles is assumed to be non-relativistic and much less then the average beam velocity V_b in the laboratory frame $(|v| \ll V_b, c$ where c is the speed of light *in vacuo*).

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The Vlasov-Poisson equations provide a fully nonlinear, self-consistent description of collective processes in the charged particle beam in the electrostatic approximation.[1,2] The Vlasov-Poisson equations are

$$\frac{\partial f_b}{\partial t}+\frac{\mathbf{p}}{m_b}\cdot\frac{\partial f_b}{\partial \mathbf{x}}+(-e_b\nabla\phi-\nabla\psi_f)\cdot\frac{\partial f_b}{\partial \mathbf{p}}=0 \tag{1}$$

and

$$\nabla^2\phi=-4\pi e_b\int d^3p f_b, \tag{2}$$

where $f_b(\mathbf{x},\mathbf{p},t)$ is the beam distribution function, ϕ is the scalar potential for the self-electric field ($\mathbf{E}=-\nabla\phi$) and all quantities are in the beam frame.

The work in this paper assumes that the beam is surrounded by a perfectly conducting, cylindrical wall at the radial location r_w. The beam is confined transversely within the conducting pipe by the applied focusing force, and perturbed beam properties are assumed to be spatially periodic in the axial direction with fundamental periodicity length L. In summary,

$$\left.\frac{\partial\phi}{\partial z}\right|_{r=r_w}=\left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=r_w}=0, \tag{3}$$

$$f_b(r,z,\theta,\mathbf{p},t)=f_b(r,z+L,\theta,\mathbf{p},t), \tag{4}$$

$$f_b(\mathbf{x},\mathbf{p},t)\big|_{r\geq r_w}=0, \tag{5}$$

where (r,θ,z) are cylindrical polar coordinates, and $r=(x^2+y^2)^{1/2}$ is the radial distance from the cylinder axis located at $r=0$. Together, these boundary conditions and the Vlasov-Poisson equations fully describe the nonlinear electrostatic dynamics of a non-relativistic, single-species, charged particle beam propagating through a conducting cylindrical pipe.

The nonlinear dynamics described by Eqs. (1)–(5) conserve the total energy per unit length U, and the generalized entropy per unit length S_G.[2] Here, the generalized entropy per unit length is defined by

$$S_G\equiv\int d^3x\int d^3p G(f_b), \tag{6}$$

where $\int d^3x...=\frac{1}{L}\int_{-L/2}^{L/2}dz\int_0^{2\pi}d\theta\int_0^{r_w}drr...$, the momentum integral is over all momentum space, and $G(f_b)$ is any smooth differentiable function of f_b that satisfies $G(f_b\to 0)=0$ with axial periodicity length L. Thus, the total energy per unit length U, classical entropy per unit length S, and line density N are all conserved quantities (independent of t), i.e.,

$$U=\int d^3x\left[\int d^3p\left(\frac{p^2}{2m_b}+\psi_f\right)f_b+\frac{|\nabla\phi|^2}{8\pi}\right]=const., \tag{7}$$

$$S=-\int d^3x\int d^3p f_b ln(f_b)=const., \tag{8}$$

$$N=\int d^3x\int d^3p f_b=const. \tag{9}$$

The generalized Helmholtz free energy F is a linear combination of these conserved quantities, defined by

$$F\equiv U-TS-\mu N, \tag{10}$$

where T and μ are constant multipliers. Explicitly, the Helmholtz free energy F can be expressed as

$$F[f_b(\mathbf{x},\mathbf{p},t)]=\int d^3x\int d^3p\left[\frac{p^2}{2m_b}+\psi_f+Tln(f_b)-\mu\right]f_b+\int d^3x\frac{|\nabla\phi|^2}{8\pi}. \tag{11}$$

As a linear combination of conserved quantities, the generalized Helmholtz free energy functional F defined in Eq. (11) is exactly conserved as the beam dynamically evolves from an initial state f_{b0} to some later state f_b. Therefore,

$$\begin{aligned}\Delta F(f_b,f_{b0})\equiv{}&F[f_b(\mathbf{x},\mathbf{p},t)]-F[f_{b0}(\mathbf{x},\mathbf{p},t_0)]\\ ={}&\int d^3x\int d^3p\left[\left(\frac{p^2}{2m_b}+\psi_f\right)(f_b-f_{b0})\right]\\ &+\frac{1}{8\pi}\int d^3x[|\nabla\phi|^2-|\nabla\phi_0|^2]+\int d^3x\int d^3p[Tf_b ln(f_b)\\ &-Tf_{b0}ln(f_{b0})-\mu(f_b-f_{b0})]=0.\end{aligned} \tag{12}$$

Equation (12) is an exact consequence of the nonlinear Vlasov-Poisson equations and the boundary conditions in Eqs. (1)–(5). Equation (12) forms the basis for calculating the nonlinear bound on electrostatic perturbations in Secs. III and IV.

III. REFERENCE EQUILIBRIUM STATE WITH MINIMUM HELMHOLTZ FREE ENERGY

In this section, we determine the equilibrium distribution that corresponding to an absolute minimum in the Helmholtz free energy. This distribution will be used as a reference state in the remainder of this paper. The extrema of the generalized Helmholtz free energy defined in Eq. (11) correspond to zeros of the variation of the functional F with respect to the distribution function f_b, i.e.,

$$\begin{aligned}\delta F&=0\\ &=\int d^3x\int d^3p[p^2/2m_b+\psi_f+e_b\phi+Tln(f_b)+T-\mu]\delta f_b.\end{aligned} \tag{13}$$

Here, Poisson's equation and the boundary conditions in Eqs. (3)–(5) have been combined as follows:

$$\begin{aligned}\delta\left(\int d^3x\frac{|\nabla\phi|^2}{8\pi}\right)&=\int d^3x\frac{\nabla\phi\cdot\nabla\delta\phi}{4\pi}=-\int d^3x\frac{\phi\cdot(\nabla^2\delta\phi)}{4\pi}\\ &=\int d^3x\int d^3p e_b\phi\delta f_b.\end{aligned} \tag{14}$$

The class of distribution functions that satisfy the condition in Eq. (13) is the class $f_b=g_b$ defined by

$$g_b(\mathbf{x},\mathbf{p}) = \beta \exp\left(-\frac{\frac{p^2}{2m_b} + \psi_f + e_b\phi_g}{T}\right), \quad (15)$$

$$\nabla^2\phi_g = -4\pi e_b \int d^3p g_b, \quad (16)$$

where the constant β is related to the constants μ and T by $\mu = T[1 + ln(\beta)]$. Here, the conditions $g_b \geq 0$ and $g_b(\mathbf{x}, |\mathbf{p}| \to \infty) = 0$ impose the requirements that T and β be positive constants. Note that the reference distribution g_b defined in Eq. (15) corresponds to the isotropic thermal equilibrium distribution with uniform temperature T. The second variation of the Helmholtz free energy evaluated at the thermal equilibrium distribution g_b can be expressed as

$$\begin{aligned}\delta^2 F\Big|_{g_b} &= \int d^3x \int d^3p \left[\delta\phi\delta f_b + \frac{T}{f_b}(\delta f_b)^2\right]_{g_b} \\ &= \int d^3x \left[\frac{(\nabla\delta\phi)^2}{4\pi} + \int d^3p \frac{T}{g_b}(\delta f_b)^2\right] > 0. \quad (17)\end{aligned}$$

Thus, the isotropic thermal equilibrium reference distribution in Eq. (15) represents the absolute minimum of the generalized Helmholtz free energy F.[1,2] This result, combined with the linear and nonlinear stability of the thermal equilibrium state,[1,2,24] motivates the special significance given to the thermal equilibrium g_b as a reference state.

IV. NONLINEAR BOUND ON THE CHANGE IN ELECTROSTATIC FIELD PERTURBATION ENERGY

The conservation of generalized Helmholtz free energy places a strict constraint on the nonlinear evolution of the energy in the field perturbations in charged particle beams, calculated here with respect to the reference distribution defined in Sec. III. In the remainder of this paper, we use the notation,

$$\delta\phi(\mathbf{x},t) = \phi(\mathbf{x},t) - \phi_g(\mathbf{x}), \quad (18)$$

$$\delta f_b(\mathbf{x},\mathbf{p},t) = f_b(\mathbf{x},\mathbf{p},t) - g_b(\mathbf{x},\mathbf{p}), \quad (19)$$

where $\delta\phi$ and δf_b denote the changes in ϕ and f_b relative to the thermal equilibrium reference state ϕ_g, and g_b defined in Eqs. (15) and (16). The self-field energy associated with the perturbation $\delta\phi = \phi - \phi_g$ is

$$\epsilon_F \equiv \frac{1}{8\pi}\int d^3x (\nabla\delta\phi)^2. \quad (20)$$

As the beam evolves in time from an initial distribution $f_{b0}(\mathbf{x},\mathbf{p},0)$ to a distribution $f_b(\mathbf{x},\mathbf{p},t)$ at time t, the change in the field perturbation energy is

$$\begin{aligned}\Delta\epsilon_F &\equiv \frac{1}{8\pi}\int d^3x[(\nabla\delta\phi)^2 - (\nabla\delta\phi_0)^2] \\ &= \frac{1}{8\pi}\int d^3x\left[(\nabla\phi)^2 - (\nabla\phi_0)^2 - 8\pi e_b\phi_g \int d^3p(f_b - f_{b0})\right], \quad (21)\end{aligned}$$

where ϕ_0 is the self-field potential associated with the initial distributions f_{b0}. Equations (12) and (21) combine to give

$$\begin{aligned}\Delta\epsilon_F(f_b,f_{b0}) = &-\int d^3x \int d^3p[(p^2/2m_b + \psi_f + e_b\phi_g)(f_b - f_{b0})] \\ &-\int d^3x \int d^3p[Tf_b ln(f_b) - Tf_{b0} ln(f_{b0}) \\ &-\mu(f_b - f_{b0})]. \quad (22)\end{aligned}$$

At extrema of $\Delta\epsilon_F$ with respect to the distribution f_b, the variation is zero. This corresponds to the condition

$$\begin{aligned}\delta(\Delta\epsilon_F) = 0 = &-\int d^3x \int d^3p[p^2/2m_b + \psi_f + e_b\phi_g \\ &+ Tln(f_b) + T - \mu]\delta f_b. \quad (23)\end{aligned}$$

The distribution function that satisfies Eq. (23) is again the solution $f_b = g_b$ defined by Eqs. (15) and (16). Again, $\mu = T[1 + ln(\beta)]$, and T and β are positive constants. The corresponding second variation of $\Delta\epsilon_F$ is negative, i.e.,

$$\delta^2\Delta\epsilon_F\Big|_{g_b} = -\int d^3x \int d^3p \frac{T}{g_b}(\delta f_b)^2 < 0. \quad (24)$$

This implies that the reference distribution g_b corresponds to a maximum in the value of $\Delta\epsilon_F$. Therefore, the energy change functional $\Delta\epsilon_F(f_b,f_{b0})$ for any initial distribution f_{b0} is bounded from above by

$$\begin{aligned}[\Delta\epsilon_F]_{max} = \Delta\epsilon_F(g_b,f_{b0}) = &\int d^3x \int d^3p\left\{\left[\frac{p^2}{2m_b} + \psi_f\right.\right. \\ &\left.\left. + e_b\phi_g - T + Tln\left(\frac{f_{b0}}{\beta}\right)\right]f_{b0} + Tg_b\right\}. \quad (25)\end{aligned}$$

Equation (25) provides an upper bound on the change in field perturbation energy between the initial distribution and any other distribution. The constants T and β that *minimize* the upper bound in Eq. (25) provide the most physically instructive bound. These constants are determined from the transcendental equations,

$$\begin{aligned}\frac{\partial[\Delta\epsilon_F]_{max}}{\partial T} = &\int d^3x \int d^3p e_b \frac{\partial\phi_g}{\partial T}(f_{b0} - g_b) \\ &- (S_0 - S_g) - (1 + ln\beta)(N_0 - N_g) = 0, \quad (26)\end{aligned}$$

and

$$\frac{\partial[\Delta\epsilon_F]_{max}}{\partial\beta} = \int d^3x \int d^3p e_b \frac{\partial\phi_g}{\partial\beta}(f_{b0} - g_b) - \frac{T}{\beta}(N_0 - N_g) = 0. \quad (27)$$

Here, S_0 and N_0 correspond to the entropy per unit length and number of particles per unit length for the distribution f_{b0}, while S_g and N_g are the similar quantities for the thermal equilibrium reference distribution g_b.

For a specified initial distribution f_{b0}, the conditions given by Eqs. (26) and (27) determine the constants T and β which minimize the upper bound in Eq. (25). The system of Eqs. (25)–(27) defines a change in field perturbation energy larger than or equal to any change that could occur as the beam evolves through any possible state consistent with Eqs. (1)–(5). This provides a fully nonlinear, *lowest upper bound* on the possible change in field perturbation energy.

The constraints given by Eqs. (26) and (27) are a highly nonlinear set of equations, which depend on the properties of the focusing potential ψ_f, the initial distribution function f_{b0}, the reference distribution g_b, and the self-field potentials ϕ_0 and ϕ_g determined from their respective Poisson's equations. The functional forms of the initial distribution function f_{b0} and focusing potential ψ_f must be specified to determine the values of the constants T and β.

V. ANALYTICAL LIMITS

This section solves analytically the two limiting cases corresponding to a space-charge-dominated beam, and an emittance-dominated beam. In both cases, the initial distribution f_{b0} is assumed to be an anisotropic bi-Maxwellian in momentum, infinite in length, and azimuthally symmetric. The transverse focusing force acting on the beam particles is assumed to be the smooth-focusing approximation of cyclical quadrupole magnets described by the effective potential,

$$\psi_f = \frac{1}{2} m_b \omega_f^2 r^2, \tag{28}$$

where ω_f is the constant transverse frequency associated with the applied focusing field.[1,2] In the space-charge-dominated regime, $N^2 e_b^2 \gg 2NT$, an initial distribution of this type corresponds to a flat-top radial density profile,

$$f_{b0}(\mathbf{x}, \mathbf{p}) = \frac{\hat{n}_0}{(2\pi m_b T_\perp^{2/3} T_\parallel^{1/3})^{3/2}} H(r_b - r) \times \exp\left(-\frac{p_\perp^2}{2m_b T_\perp} - \frac{p_\parallel^2}{2m_b T_\parallel}\right), \tag{29}$$

where H is the Heaviside step function, $T_\perp$ and $T_\parallel$ are the transverse and longitudinal temperatures, respectively, $\hat{n}_0$ is a constant associated with the on-axis density, and r_b is the beam edge radius.[1,2] This choice is conveniently similar in form to the reference distribution g_b in the space-charge-dominated limit. In the limit $N^2 e_b^2 \gg 2NT$, the reference distribution reduces to

$$g_b(\mathbf{x}, \mathbf{p}) = \beta H(r_g - r) \exp\left(-\frac{p^2}{2m_b T}\right), \tag{30}$$

and the corresponding self-field potential is

$$\phi_g(r) = \begin{cases} -e_b N_g r^2 / r_g^2, & r \le r_g, \\ -e_b N_g \left[1 + 2ln\left(\frac{r}{r_g}\right)\right], & r > r_g. \end{cases} \tag{31}$$

Here, r_g is the thermal equilibrium distribution edge radius in the space-charge-dominated limit, and $N_g = \pi r_g^2 \beta \times (2\pi m_b T)^{3/2}$. Substituting this reference potential into Eqs. (26) and (27) gives

$$\frac{\partial [\Delta\epsilon_F]_{max}}{\partial \beta} = 0 = \frac{1}{2} N_g e_b^2 \left(\frac{r_b^2}{r_g^2} N_0 - N_g\right) - T(N_0 - N_g) \tag{32}$$

and

$$\frac{\partial [\Delta\epsilon_F]_{max}}{\partial T} = 0 = -\frac{3}{2} \frac{N_g e_b^2}{2T} \left(\frac{r_b^2}{r_g^2} N_0 - N_g\right) + S_0 - S_g - (1 - ln\beta)(N_0 - N_g). \tag{33}$$

Enforcing radial force balance on both distributions in this limit gives

$$e_b \frac{\partial \phi_0}{\partial r} = \frac{\partial \psi_f}{\partial r} = e_b \frac{\partial \phi_g}{\partial r}, \tag{34}$$

for $r \le r_b, r_g$, which combines with Poisson's equation to give

$$\frac{N_0}{N_g} = \frac{r_b^2}{r_g^2}. \tag{35}$$

This reduces the constraint conditions (26) and (27) determining T and β to the simple requirements $S_0 = S_g$ and $N_0 = N_g$. Enforcing these two constraints, a straightforward calculation gives the corresponding nonlinear bound in the simplified form,

$$[\Delta\epsilon_F]_{max} = N_0 T_\perp \left[1 - \frac{3}{2}\left(\frac{T_\parallel}{T_\perp}\right)^{1/3} + \frac{1}{2}\left(\frac{T_\parallel}{T_\perp}\right)\right]. \tag{36}$$

The expression for $[\Delta\epsilon_F]_{max}$ given in Eq. (36) is similar to that given in Ref. 25 for a general bi-Maxwellian momentum distribution in a uniform-density neutral plasma. This is consistent with the fact that the transverse focusing force in this limit cancels the self-field force, and leaves a purely kinetic Hamiltonian. Although the initial distributions are assumed to be azimuthally symmetric, the bound does not require that the perturbations have any symmetry beyond the periodicity condition in Eq. (4). All bounds in this paper apply to any electrostatic perturbation accessible within the fully nonlinear Vlasov-Poisson equations.

The opposite limit, $2NT_\perp \gg N^2 e_b^2$, corresponds to emittance-dominated beams in which space-charge forces are negligibly small to leading order. The thermal equilibrium distribution in this limit becomes

$$g_b(\mathbf{x}, \mathbf{p}) = \beta \exp\left(-\frac{p^2 + m_b^2 \omega_f^2 r^2}{2m_b T}\right). \tag{37}$$

Again choosing a similar form to the thermal equilibrium distribution, the anisotropic initial distribution f_{b0} is taken to be

$$f_{b0}(\mathbf{x}, \mathbf{p}) = \frac{\hat{n}_0}{(2\pi m_b T_\perp^{2/3} T_\parallel^{1/3})^{3/2}} \exp\left(-\frac{p_\perp^2 + m_b^2 \omega_f^2 r^2}{2m_b T_\perp} - \frac{p_\parallel^2}{2m_b T_\parallel}\right). \tag{38}$$

In this limit, the electric self-field potential and its variation with the constants T and β are negligibly small. This again gives the conditions $S_0 = S_g$ and $N_0 = N_g$, where S_0 and N_0 now correspond to the entropy per unit length and number of particles per unit length of the initial distribution in Eq. (38). Determining T and β from these conditions, and applying the force balance condition $\psi_s(r_b) = T_\perp$, the field perturbation energy bound simplifies to become

$$[\Delta\epsilon_F]_{max} = N_0 T_\perp \left[2 - \frac{5}{2}\left(\frac{T_\parallel}{T_\perp}\right)^{1/5} + \frac{1}{2}\left(\frac{T_\parallel}{T_\perp}\right)\right]. \tag{39}$$

There is no *a priori* reason that the field perturbations will grow to levels near the lowest upper bound estimates in Eqs. (36) and (39). The common behavior of the two expressions, however, gives insight into the physical relevance of this thermodynamic bound on the change in field perturbation energy. For anisotropies $0 \le T_\parallel/T_\perp \le 1$, both bounds are maximized when $T_\parallel = 0$. For the space-charge-dominated case, this maximum is the transverse kinetic energy $N_0 T_\perp$ and represents the extreme case where all of the transverse kinetic energy of the beam particles is available to drive unstable perturbations. The emittance-dominated maximum is larger, allowing for both the kinetic energy and the focusing potential energy as possible sources of free energy. The bounds drop rapidly from these maximum values as $T_\parallel/T_\perp$ increases from 0 to 1, even as the total kinetic energy of the initial distributions increases. This steep drop-off represents a significant improvement in understanding the free energy available to increase the energy in field perturbations. It may also suggest that the bound is approaching actual attainable values, making it a relevant guide for experimental expectations. Both bounds clearly show, for example, that the beam is stable for the case where $T_\parallel/T_\perp = 1$. This is consistent with the known linear and nonlinear stability of an isotropic Maxwellian distribution.[1,2,24]

Figure 1 shows the two analytical bounds given in Eqs. (36) and (39) alongside numerical solutions for the generalized bi-Maxwellian distribution given by

$$f_{b0}(\mathbf{x},\mathbf{p}) = \frac{\hat{n}_0}{(2\pi m_b T_\perp^{2/3} T_\parallel^{1/3})^{3/2}} \times \exp\left[-\frac{p_\perp^2 + m_b^2\omega_f^2 r^2 + 2m_b e_b \phi_0(r)}{2m_b T_\perp} - \frac{p_\parallel^2}{2m_b T_\parallel}\right], \tag{40}$$

which approaches the profiles given by Eqs. (29) and (38) in the space-charge-dominated and emittance-dominated limits. The results in Fig. 1 not taken from analytical limits were obtained by numerical optimization of the free parameters T and β to minimize the upper bound given by Eq. (25). All of the solutions exhibit the same general behavior, and the bound smoothly transitions from one limiting analytical solution to the other. Each curve shows the simple, yet powerful physics encapsulated in the fully nonlinear thermodynamic bound.

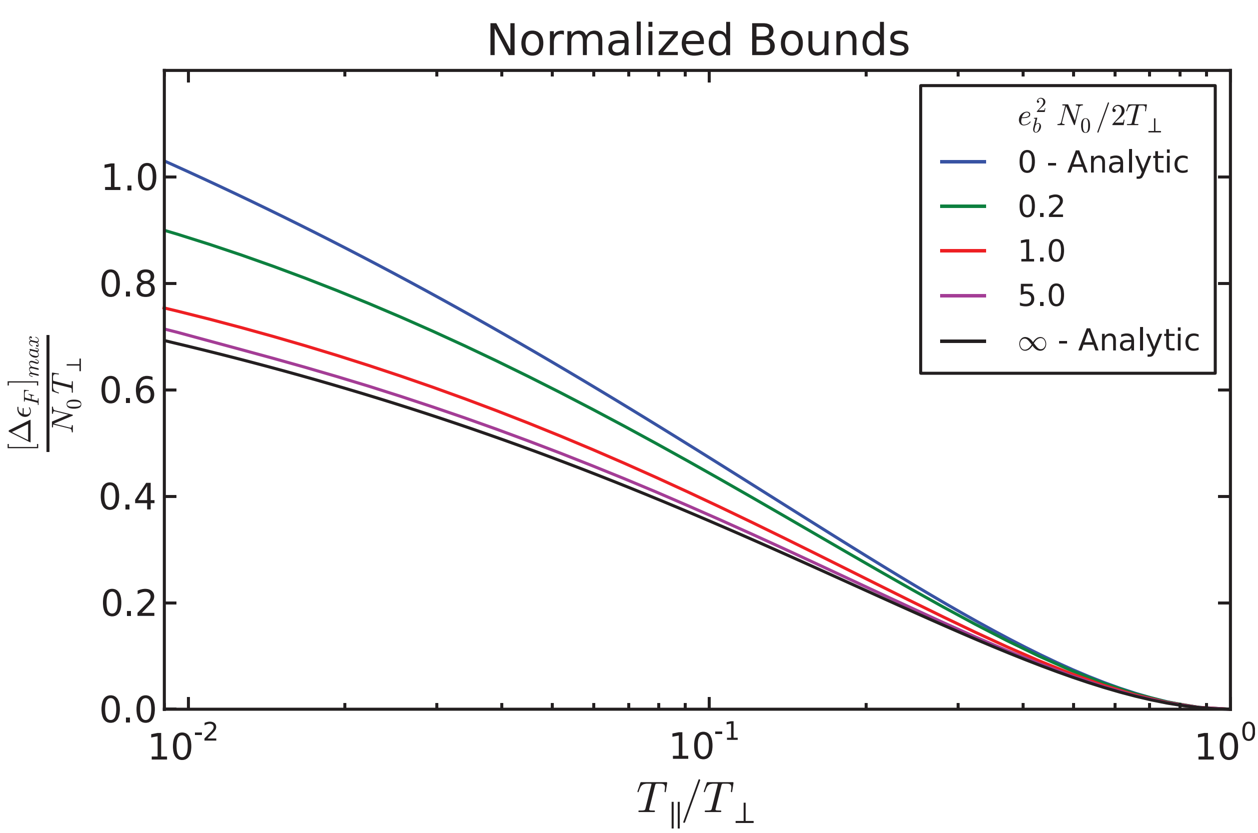

FIG. 1. The normalized nonlinear thermodynamic bound on the change in field perturbation energy as a function of $T_\parallel/T_\perp$ for bi-Maxwellian distributions described by Eq. (40) for several values of the dimensionless parameter $e_b^2 N_0/2T$. The bounding curves in the emittance-dominated limit ($e_b^2 N_0/2T \to 0$) and space-charge-dominated limit ($e_b^2 N_0/2T \to \infty$) are obtained from the analytical estimates in Eqs. (39) and (36), respectively.

VI. CONCLUSIONS

The main result of this paper is that the lowest upper bound on the unstable field energy of electrostatic perturbations in a single-species charged particle beam is given by Eq. (25) with the appropriate constant multipliers T and β determined from Eqs. (26) and (27). This bound is a result of the conservation of generalized Helmholtz free energy and applies to all electrostatic perturbations. The bound is a strong statement about the possible severity of nonlinear instabilities such as Harris-type instabilities that develop in intense anisotropic non-neutral beams. The lowest upper bound on the field perturbation energy of these instabilities fully encompasses intense self-field and nonlinear effects while making no assumption as to the detailed structure of the perturbations. The two analytical examples presented here demonstrate that this bound can significantly improve our understanding of the free energy available to drive these instabilities. Finally, the formalism developed here can be applied for any specified initial distribution function f_{b0} and provides a powerful framework for determining the lowest upper bound on the field energy associated with unstable electrostatic perturbations.

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