# A class of generalized Kapchinskij-Vladimirskij solutions and associated envelope equations for high-intensity charged particle beams

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## Abstract

A class of generalized Kapchinskij-Vladimirskij solutions of the nonlinear Vlasov-Maxwell equations and the associated envelope equations for high-intensity beams in a periodic lattice is derived. It includes the classical Kapchinskij-Vladimirskij solution as a special case. For a given lattice, the distribution functions and the envelope equations are specified by eight free parameters. The class of solutions derived captures a wider range of dynamical envelope behavior for high-intensity beams, and thus provides a new theoretical tool to investigate the dynamics of high-intensity beams.

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For high-intensity charged particle beams in an uncoupled periodic transverse focusing lattice, the beam envelope dynamics described by the envelope equations is an important research topic for optimizing beam quality and controlling beam instability. The most comprehensive self-consistent description of high-intensity beam dynamics, including both collective transverse dynamics [1–3] and longitudinal dynamics[4], is a kinetic description using the nonlinear Vlasov-Maxwell (VM) equations [5]. In 1959, Kapchinskij and Vladimirskij [5, 6] derived the envelope equations as a rigorous solution of the VM equations for a special distribution function, which is now called the Kapchinskij-Vladimirskij (KV) distribution. Since then, the envelope equations have become a very important theoretical tool for investigating the transverse dynamics of high-intensity beams in uncoupled periodic focusing lattices [1–3, 7–15].

In this Letter, we derive a class of generalized Kapchinskij-Vladimirskij solutions of the VM equations and the associated nonlinear envelope equations for high-intensity beams in an uncoupled periodic transverse focusing lattice. The new class of distribution functions and the associated envelope equations include the classical KV distribution function and the associated envelope equations as a special case. In the classical KV solution, for a given focusing lattice and a line density of the beam, the distribution function and associated envelope equations are specified by two free parameters, *i.e.*, the transverse emittances $\varepsilon_x$ and $\varepsilon_y$. In the generalized solutions described in this letter, the distribution functions and associated envelope equations are specified by eight free parameters, *i.e.*, two transverse emittances $\varepsilon_x$ and $\varepsilon_y$, and two $2\times 2$ symmetric and positive definite matrices $\xi_x$ and $\xi_y$. The KV solution is a special case of the solutions presented here when $\xi_x = \xi_y = I$, where $I$ is the $2\times 2$ unit matrix. The choice of $\xi_x$ and $\xi_y$ other than the unit matrix $I$ introduces the dependence on the phase advance $\phi_x$ and $\phi_y$. Therefore, the new set of envelope equations enable us to study a much wider class of beam envelope dynamics.

Our starting point is the Vlasov-Maxwell equations that govern the nonlinear evolution of the distribution function $f$ and the normalized self-field potential $\psi$,

$$\frac{\partial f}{\partial s} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - (\nabla\psi + \kappa_{qx} x \boldsymbol{e}_x + \kappa_{qx} y \boldsymbol{e}_y)\cdot\frac{\partial f}{\partial \mathbf{v}} = 0\,, \tag{1}$$

$$\nabla^2\psi = \frac{-2\pi K_b}{N_b}\int f dv_x dv_y\,. \tag{2}$$

Here, the normalized self-field potential is defined by $\psi = q_b\phi/\gamma_b^3 m\beta_b^2 c^2$, where $\phi$ is the space-charge potential, $\beta_b c$ is the directed beam velocity in the longitudinal direction,

$\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, $s = \beta_b ct$ is an effective time variable normalized by $1/\beta_b c$, $K_b = 2N_b q_b^2/\gamma_b^3 m\beta_b^2 c^2$ is the beam self-field perveance, and $N_b = \int f dxdydv_x dv_y$ is the line density. Particle motion in the beam frame is assumed to be non-relativistic, and $(x, y)$ is the transverse displacement of a beam particle, $\mathbf{v} = d\mathbf{x}/ds = (v_x, v_y)$ is the normalized transverse velocity in the beam frame, and $\kappa_{qx}$ and $\kappa_{qy}$ are the focusing coefficients for the quadrupole lattice. The $-\nabla\psi$ term in Eq. (1) describes the self-field force due to the self-electric and self-magnetic fields of the beam, and it is nonlinearly coupled to the distribution function $f$ through Eq. (2). Equations (1) and (2) form a nonlinear integro-differential equation system, and it is in general difficult to find analytical solutions.

Kapchinskij and Vladimirskij [5, 6] discovered a remarkable solution of the nonlinear VM equations (1) and (2), which is now called the KV distribution. The solution is constructed from the well-known Courant-Snyder invariants [16] for a linear focusing lattice

$$I_x = \frac{x^2}{w_x^2} + \left(w_x\dot{x} - x\dot{w}_x\right)^2 \ , \ I_y = \frac{y^2}{w_y^2} + \left(w_y\dot{y} - y\dot{w}_y\right)^2 . \tag{3}$$

Here, $\varepsilon_x$ and $\varepsilon_y$ are the constant transverse emittances, and $w_x$ and $w_y$ are the envelope functions satisfying the envelope equations,

$$\ddot{w}_x + (\kappa_{qx} + \kappa_{sx})w_x = w_x^{-3}\ , \ \ddot{w}_y + (\kappa_{qy} + \kappa_{sy})w_y = w_y^{-3} . \tag{4}$$

In Eq. (4), the self-field force are assumed *a prior* to be a linear function of the displacement with the defocusing coefficient $\kappa_{sx}$ and $\kappa_{sy}$, *i.e.*, $-\nabla\psi = -\kappa_{sx}x\boldsymbol{e}_x - \kappa_{sy}y\boldsymbol{e}_y$. The coefficients $\kappa_{sx}$ and $\kappa_{sy}$ will be determined self-consistently from the distribution function, which is required to satisfy the Vlasov equation (1) and simultaneously generate a linear self-field force in order for the CS invariants to be valid. A distribution function that satisfies both conditions is the KV distribution given by

$$f_{KV} = \frac{N_b}{\pi\varepsilon_x\varepsilon_y}\delta\left(\frac{I_x}{\varepsilon_x} + \frac{I_y}{\varepsilon_y} - 1\right) , \tag{5}$$

which obviously satisfies the Vlasov equation (1) because it is a function of the invariants of the particle dynamics. Here, the constants $\varepsilon_x$ and $\varepsilon_y$ are the transverse emittances. The density profile projected by the distribution function $f_{KV}$ in the transverse configuration space is

$$n\left(x, y, s\right) = \int d\dot{x}d\dot{y}f_{KV} = \begin{cases} N_b/\pi ab,\ 0 \le x^2/a^2 + y^2/b^2 < 1\,, \\ 0,\ 1 < x^2/a^2 + y^2/b^2\,. \end{cases} \tag{6}$$

where $a \equiv \sqrt{\varepsilon_x} w_x$ , $b \equiv \sqrt{\varepsilon_y} w_y$ . This density profile corresponds to a constant-density beam with elliptical cross-section and pulsating transverse dimensions $a$ and $b$. The associated normalized self-field inside the beam, determined from Eq. (2), is given by

$$\psi = \frac{-K_b}{a+b}\left(\frac{x^2}{a}+\frac{y^2}{b}\right),\; 0 \le x^2/a^2 + y^2/b^2 < 1, \tag{7}$$

which indeed generates a linear defocusing force with coefficients $\kappa_{sx} = -2K_b/a(a+b)$, and $\kappa_{sy} = -2K_b/b(a+b)$. The KV solution reduces the nonlinear VM equations to the envelope equations given by Eq. (4) in terms of $w_x$ and $w_y$, or equivalently, in terms of $a$ and $b$ as

$$\ddot{a} + \kappa_{qx} a - \frac{2K_b}{(a+b)} = \frac{\varepsilon_x^2}{a^3}\,,\; \ddot{b} + \kappa_{qy} b - \frac{2K_b}{(a+b)} = \frac{\varepsilon_y^2}{b^3}\,. \tag{8}$$

The envelope equations have become an indispensable tool for our understanding of the dynamical behavior of high-intensity beams.

We now show how to construct a class of more general solutions of the nonlinear VM equations and the associated envelope equations, which include the classical KV solution as a special case. It turns out that the class of distribution functions that generate a linear space-charged force and satisfy the Vlasov equation is much wider than the KV distribution given by Eq. (5). First, we construct the following invariants for the transverse dynamics of a single particle that is more general than the CS invariants give by (3). For the dynamics in the $x-$direction, the invariant is

$$I_{\xi_x} = (x, \dot{x}) E_x^T P_x^T \xi_x P_x E_x (x, \dot{x})^T\,, \tag{9}$$

where $\xi_x$, $E_x$, and $P_x$ are $2\times 2$ matrices, and superscript "$T$" denotes matrix transpose. The matrix $\xi_x = \begin{pmatrix} \xi_{x1} & \xi_{x2} \\ \xi_{x2} & \xi_{x4} \end{pmatrix}$ is a symmetric, positive definite constant matrix, and $E_x$ and $P_x$ are defined as

$$E_x \equiv \begin{pmatrix} w_x^{-1T} & 0 \\ -\dot{w}_x & w_x \end{pmatrix}\,,\; P_x \equiv \begin{pmatrix} \cos\phi_x & -\sin\phi_x \\ \sin\phi_x & \cos\phi_x \end{pmatrix}\,, \tag{10}$$

where $w_x$ and $\phi_x$ are the envelope function and phase advance, respectively, satisfying

$$\ddot{w}_x + (\kappa_{qx} + \kappa_{sx}) w_x = w_x^{-3}\,, \tag{11}$$

$$d\phi_x/ds = 1/w_x^2\,. \tag{12}$$

For present purpose, the self-field force is assumed *a prior* to be a linear function of $x$, *i.e.*, $-\partial\psi/\partial x = -\kappa_{sx} x \boldsymbol{e}_x$. It will be determined later self-consistently from the distribution function. The superscript "$T$" denotes matrix transpose. There are several methods to verify that $I_{\xi_x}$ is a constant of the motion. The simplest approach is to exam the transfer map [17] between the initial point $(x_0, \dot{x}_0)$ and $(x, \dot{x})$, which is constructed as

$$M = E_x^{-1} P_x^{-1} E_{x0} = \begin{pmatrix} w_x & 0 \\ \dot{w}_x & \dfrac{1}{w_x} \end{pmatrix} \begin{pmatrix} \cos\phi_x & \sin\phi_x \\ -\sin\phi_x & \cos\phi_x \end{pmatrix} \begin{pmatrix} w_{x0}^{-1} & 0 \\ -\dot{w}_{x0} & w_{x0} \end{pmatrix} .$$

This implies that

$$P_x E_x \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = E_{x0} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}$$

is a constant of the motion. Therefore, $I_{\xi_x}$ given by Eq. (9) for a constant matrix $\xi_x$ is a constant of the motion. In addition, because $\xi_x$ is chosen to be symmetric and positive definite, $I_{\xi_x}$ is also positive definite.

When $I_{\xi_x}$ is chosen to be the $2 \times 2$ unit matrix $I$, the term $P_x^T \xi_x P_x$ in Eq. (9) reduces to the unit matrix $I$, and the $I_{\xi_x}$ invariant becomes the classical CS invariant given by Eq. (3). In this special case, the invariant is a function of $w_x$, $\dot{w}_x$, $x$, and $\dot{x}$ only, and does not depend on the phase advance $\phi_x$.

When the $\xi_x$ matrix is chosen to be any symmetric, positive definite matrix other than the unit matrix, the invariant $I_{\xi_x}$ will depend on the phase advance $\phi_x$. Similarly, the corresponding invariant $I_{\xi_y}$ in the $y-$direction can be constructed by replacing the subscript "$x$" with "$y$" in Eqs. (9)-(12). The distribution function for the new class of solutions is chosen to be

$$f = \frac{N_b |\xi_x| |\xi_y|}{\pi \varepsilon_x \varepsilon_y} \delta \left( \frac{I_{\xi_x}}{\varepsilon_x} + \frac{I_{\xi_y}}{\varepsilon_y} - 1 \right) , \tag{13}$$

where the constants $\varepsilon_x$ and $\varepsilon_y$ are the transverse emittances, and $|\xi_x|$ and $|\xi_y|$ are the determinants of $\xi_x$ and $\xi_y$ matrices. To be consistent with the assumption that the space-charge force is linear, it is necessary to verify that this distribution function indeed generates a linear space-charge force. For this purpose, we calculate the number density in configuration space defined by $n(x, y, s) = \int d\dot{x} d\dot{y} f$. The velocity integral here is much more difficult to calculate than in the classical KV case, because ($I_{\xi_x}$,$I_{\xi_y}$) depend on $(w_x, w_y)$ and $(\phi_x, \phi_y)$ in a non-trivial manner. The special technique required here is the Cholesky decomposition. For a symmetric, positive definite matrix $M$, it is always possible to uniquely decompose it into the form

$$M = L^T L ,$$

where $L$ is a lower triangular matrix. This is the Cholesky decomposition. In the present case, the matrix product $E_x^T P_x^T \xi_x P_x E_x$ is symmetric and positive definite because $\xi_x$ is symmetric and positive definite, and its Cholesky decomposition is

$$E_x^T P_x^T \xi_x P_x E_x = L^T L \,, \tag{14}$$

where

$$L = \begin{pmatrix} \dfrac{1}{w_x}\sqrt{\dfrac{|\xi_x|}{h_x}} & 0 \\ \dfrac{g_x}{w_x\sqrt{h_x}} - \sqrt{h_x}\dot{w}_x & \sqrt{h_x}w_x \end{pmatrix}, \tag{15}$$

$$h_x = \xi_{x4}\cos^2\phi_x + \xi_{x1}\sin^2\phi_x - \xi_{x2}\sin 2\phi_x \,, \tag{16}$$

$$g_x = (\xi_{x4} - \xi_{x1})\cos\phi_x \sin\phi_x + \xi_{x2}\cos 2\phi_x \,. \tag{17}$$

Equation (14) can be verified by straightforward calculation. It is straightforward to confirm that $h_x$ is positive definite from the fact that $\xi_x$ is positive definite. With help of this decomposition, the $I_{\xi_x}$ invariants can be expressed as

$$I_{\xi_x} = \left(\frac{x}{w_x}\sqrt{\frac{|\xi_x|}{h_x}}\right)^2 + \left[\left(\frac{g_x}{w_x\sqrt{h_x}} - \sqrt{h_x}\dot{w}_x\right)x + \sqrt{h_x}w_x\dot{x}\right]^2 . \tag{18}$$

The $I_{\xi_y}$ invariant has the same function form, with the subscript "$x$" replaced by the subscript "$y$" in Eqs. (14)-(18).

With the following change of variables

$$V_x = \left(\frac{g_x}{w_x\sqrt{h_x}} - \sqrt{h_x}\dot{w}_x\right)\frac{x}{\sqrt{\varepsilon_x}} + \frac{\sqrt{h_x}}{\sqrt{\varepsilon_x}}w_x\dot{x} \,,$$
$$V_y = \left(\frac{g_y}{w_y\sqrt{h_y}} - \sqrt{h_y}\dot{w}_y\right)\frac{y}{\sqrt{\varepsilon_y}} + \frac{\sqrt{h_y}}{\sqrt{\varepsilon_y}}w_y\dot{y} \,,$$
$$d\dot{x}d\dot{y} = \frac{1}{\sqrt{\varepsilon_x\varepsilon_y}w_x w_y}dV_x dV_y \,,$$

the velocity integral $\int d\dot{x}d\dot{y}f$ can now be carried out exactly to give

$$\begin{aligned} n(x,y,s) &= \int d\dot{x}d\dot{y}f \\ &= \int dV_x dV_y \frac{N_b}{\pi w_x w_y}\sqrt{\frac{|\xi_x||\xi_y|}{h_x h_y \varepsilon_x \varepsilon_y}}\,\delta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + V_x^2 + V_y^2 - 1\right) \\ &= \begin{cases} \dfrac{N_b}{\pi ab}, \; 0 \le x^2/a^2 + y^2/b^2 < 1 \,, \\ 0, \; 1 < x^2/a^2 + y^2/b^2 \,, \end{cases} \end{aligned}$$

where $a$ and $b$ are the transverse dimension of the ellipse defined by $a \equiv w_x\sqrt{h_x\varepsilon_x/|\xi_x|}$ and $b \equiv w_y\sqrt{h_y\varepsilon_y/|\xi_y|}$. The corresponding effective temperature profile is calculated to be

$$
\begin{aligned}
T_\perp(x,y,s) &\equiv \left\langle (\dot{x}-\langle\dot{x}\rangle)^2 + (\dot{y}-\langle y\rangle)^2 \right\rangle \\
&= \begin{cases} \left(\dfrac{\varepsilon_x}{2h_x w_x^2} + \dfrac{\varepsilon_y}{2h_y w_y^2}\right)\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right) & 0 \le x^2/a^2 + y^2/b^2 < 1\,, \\ 0, & 1 < x^2/a^2 + y^2/b^2\,, \end{cases}
\end{aligned}
$$

where $\langle\chi\rangle \equiv \int \chi f d\dot{x}d\dot{y} / \int f d\dot{x}d\dot{y}$ denotes the moment of $\chi$. As in the case of the classical KV solution, the density profile $n(x,y,s)$ corresponds to a constant-density beam with elliptical cross-section and pulsating transverse dimensions $a$ and $b$. The normalized self-field potential $\psi$ is quadratic inside the beam [see Eq. (7)], and the corresponding self-field force is linear with $\kappa_{sx} = -2K_b/a(a+b)$, $\kappa_{sy} = -2K_b/b(a+b)$. The difference here is that the transverse dimensions $a$ and $b$ now depend on the phase advance $\phi_x$ and $\phi_y$ through the functions $h_x$ and $h_y$. Thus, we have completed the construction of the new class of solution of the VM equations and the associated envelope equation.

To summarize, the class of generalized KV solutions is specified by the distribution function in Eq. (13), which reduces the VM equations to a set of envelope equations for $w_x$, $w_y$, $\phi_x$, and $\phi_y$. For easy reference, the complete set of envelope equations is listed here,

$$\ddot{w}_x + \left[\kappa_{qx} - \frac{2K_b}{a(a+b)}\right] w_x = w_x^{-3}\,, \tag{19}$$

$$\ddot{w}_y + \left[\kappa_{qy} - \frac{2K_b}{b(a+b)}\right] w_y = w_x^{-3}\,, \tag{20}$$

$$\dot{\phi}_x = 1/w_x^2\,,\ \dot{\phi}_y = 1/w_y^2\,, \tag{21}$$

$$a \equiv w_x\sqrt{h_x\varepsilon_x/|\xi_x|}\,,\ b \equiv w_y\sqrt{h_y\varepsilon_y/|\xi_y|}\,, \tag{22}$$

$$h_x \equiv \xi_{x4}\cos^2\phi_x + \xi_{x1}\sin^2\phi_x - \xi_{x2}\sin 2\phi_x\,, \tag{23}$$

$$h_y \equiv \xi_{y4}\cos^2\phi_y + \xi_{y1}\sin^2\phi_y - \xi_{y2}\sin 2\phi_y\,, \tag{24}$$

where the constants $\varepsilon_x$ and $\varepsilon_y$ are the transverse emittances, and $\xi_x = \begin{pmatrix} \xi_{x1} & \xi_{x2} \\ \xi_{x2} & \xi_{x4} \end{pmatrix}$ and $\xi_y = \begin{pmatrix} \xi_{y1} & \xi_{y2} \\ \xi_{y2} & \xi_{y4} \end{pmatrix}$ are symmetric, positive definite constant matrices.

For a given periodic lattice and beam line density $N_b$, the distribution function and the envelope equations are specified by eight free parameters, *i.e.*, $\varepsilon_x$, $\varepsilon_y$, $\xi_x$, and $\xi_y$ . This class of solutions includes the KV distribution and associated envelope equations as a special case when $\xi_x = \xi_y = I$. This class of solutions is also more general than the classical KV solution because when $\xi_x \neq I$ and/or $\xi_y \neq I$, the envelope functions $w_x$ and $w_y$ are coupled to the phase advance $\phi_x$ and $\phi_y$ through $h_x$ and $h_y$, whereas for the KV solution $w_x$and $w_y$ are decoupled from $\phi_x$ and $\phi_y$. Thus, only two free parameters, *i.e.*, $\varepsilon_x$ and $\varepsilon_y$, are needed to specify the classical KV solution. Because of the extra degree of freedom introduced by the matrices $\xi_x$ and $\xi_y$ , the new class of solutions derived here can capture a wider range of dynamical envelope behavior for high-intensity beams. As a result of the dependence on the phase advance, $\phi_x$ and $\phi_y$, the new set of envelope equations in general do not admit solutions matched to the periodic lattice, since the phase advance is in general not a integer fraction of $2\pi$.

We now give two numerical examples of the envelope dynamics described by the new set of envelope equations. We consider the case of a high-intensity beam with $\varepsilon_x = \varepsilon_y = \varepsilon$ and normalized self-field perveance is $K_b/\varepsilon = 10$ in a FODO (acronym for focusing-off-defocusing-off) focusing lattice with normalized quadrupole focusing field amplitude $\hat{\kappa}_q S \equiv q_b B'_q/\gamma_b m \beta_b c^2 = 15$ and filling factor $\eta = 0.30$, where $S$ is the lattice period. For comparison, the matched solution of the classical KV envelope equations (8) is plotted in Fig. 1 for 20 lattice periods. In Fig. 2, the envelope dynamics is shown for the case where $\xi_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi_y = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$. The nonlinear evolution of the envelope shows a complicated pattern, compared with the matched KV envelope in Fig. 2. Even though the envelope is not matched with period $S$ in Fig. 2, on a longer distance-scale, the envelope in Fig. 2 is approximately matched with period of $28S$, where $S$ denotes one lattice period. Figure 3 shows the envelope dynamics for the case where $\xi_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi_y = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$, which also manifests the feature of approximate long-distance periodicity. The approximate periodicity length of the envelope dynamics in Fig. 3 is $26S$. However, the small-scale variation and irregularity superimposed on top of the long-term periodicity is much more prominent in Fig. 3 than the case shown in Fig. 2.

In conclusion, we have derived a class of generalized KV solutions of the nonlinear VM equations and the associated envelope equations for high-intensity beams in a periodic lat-

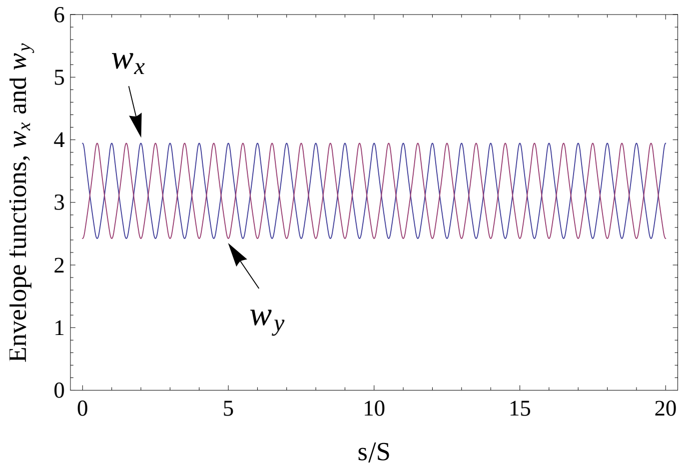


Figure 1: Envelope dynamics of a matched KV solution over the interval $0 \leq s/S \leq 20$.

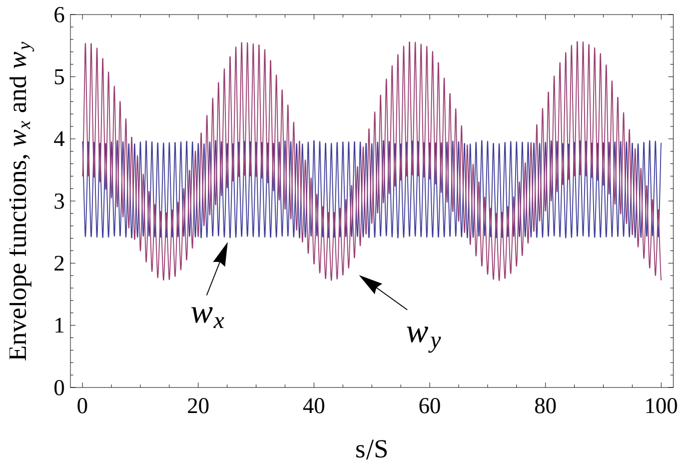


Figure 2: Envelope dynamics for the case where $\xi_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi_y = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$ over the interval $0 \leq s/S \leq 100$.

tice. It includes the classical KV solution as a special case. For a given periodic lattice and beam line density, the distribution function and the envelope equations are specified by eight free parameters. The class of solutions derived here captures a wider range of envelope dynamics for high-intensity beams, and thus provides us with a new theoretical tool to investigate the dynamics of high-intensity beams in an uncoupled periodic transverse focusing lattice.

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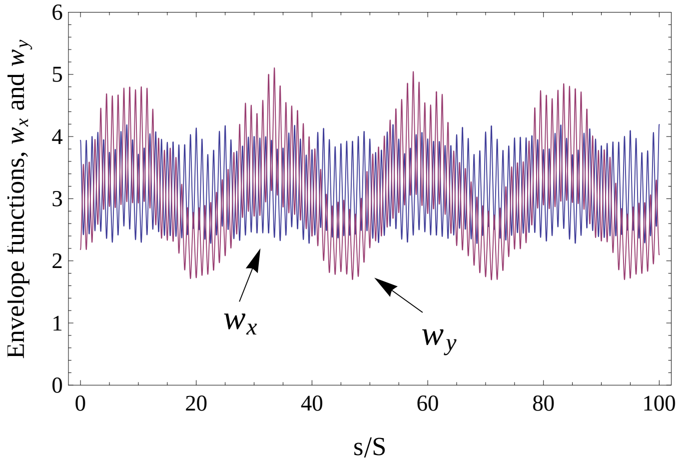


Figure 3: Envelope dynamics for the case where $\xi_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi_y = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ over the interval $0 \leq s/S \leq 100$.

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