Self-similar nonlinear dynamical solutions for one-component nonneutral plasma in a time-dependent linear focusing field

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In a linear trap confining a one-component nonneutral plasma, the external focusing force is a linear function of the configuration coordinates and/or the velocity coordinates. Linear traps include the classical Paul trap and the Penning trap, as well as the newly proposed rotating-radio-frequency traps and the Mobius accelerator. This paper describes a class of self-similar nonlinear solutions of nonneutral plasma in general time-dependent linear focusing devices, with self-consistent electrostatic field. This class of nonlinear solutions includes many known solutions as special cases. © *2011 American Institute of Physics*. [doi:10.1063/1.3600623]

Nonneutral plasmas are often confined in traps with external focusing fields, such as the Paul trap[1,2] and the Penning trap.[3–8] For most of the often-used traps, the external focusing forces are linear functions of the configuration coordinates and/or the velocity coordinates. For example, in a Paul trap the external focusing force in the transverse direction is proportional to the transverse displacement from the trap axis, whereas in a Penning trap the transverse focusing force is proportional to the transverse velocity. (Strictly speaking, this linearity is valid near the axis for real devices.) We call these types of traps *linear* traps. The strength of the external focusing field is generally allowed to vary with time. In many cases, such as the Paul trap, it is necessary to have time-dependent focusing fields to provide transverse confinement. Obviously, the quadrupole and solenoidal focusing lattices in particle accelerators[9] are also linear focusing devices. Recently, new types of traps, such as rotating-radio-frequency ion traps,[10] and new types of focusing lattices, such as the Mobius accelerator,[11] have been proposed. The main feature of these new devices is that the focusing force components in different directions are linearly coupled, which offers advantages in terms of stability and focusing strength over standard traps and focusing lattices. Yet, they all fit into the category of *linear* focusing devices. In this paper, we describe a class of self-similar nonlinear dynamical solutions of nonneutral plasmas in general linear focusing devices, with self-consistent electrostatic potential generated by the oscillating one-component plasma. The starting point of the present study is the set of macroscopic fluid equations with self-consistent electric field, which model the nonlinear dynamics of nonneutral plasmas. The class of nonlinear dynamical solutions admitted by the fluid equations includes many of the known modes in linear focusing systems as special cases, such as the well-known transverse envelope oscillations of a charged particle bunch in focusing lattices of accelerators and store rings, and the equilibrium solution of a cold nonneutral plasma in a time-independent Penning trap. It also includes new collective oscillation modes that have not been reported before. As an example, a nonlinear collective oscillating mode in a time-dependent Penning trap is identified.

Collective dynamics of nonneutral plasmas are of considerable practical importance.[2,12–20] Previous approaches for investigating collective dynamics of a nonneutral plasma typically first find an equilibrium solution, then analyze the evolution of linear perturbations relative to the equilibrium. An excellent example is the linear eigenmode analysis by Dubin[13,14,16,17] and Bollinger *et al.*.[2] Dubin[14] also developed a class of self-similar solution for the penning trap [detailed discussion is given after Eq. (11)]. Compared with the classical studies, the analysis presented here examines a new class of nonlinear modes that are applicable to time-dependent linear focusing devices, including Paul traps, and Penning traps with time-dependent confining magnetic field $B_0(t)\mathbf{e}_z$, where there exists no quasi-steady equilibrium state ($\partial/\partial t = 0$) for the plasma. It also applies to the periodic focusing lattice with quadrupole and solenoidal magnets in accelerators and storage rings.

We model the dynamics of a one-component nonneutral plasma in an applied linear focusing field including the effects of the self-generated electrostatic field, $\mathbf{E} = -\nabla\varphi$, by the following set of macroscopic fluid equations:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0\,, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{q}{m}\nabla\varphi + \frac{\nabla P}{mn} + \kappa_1(t)\cdot\mathbf{x} + \kappa_2(t)\cdot\mathbf{v} = \mathbf{0}\,, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(\frac{P}{n^{\gamma}}\right) = 0\,, \tag{3}$$

$$\nabla^2\varphi = -4\pi q n\,, \tag{4}$$

where $-\kappa_1(t)\cdot\mathbf{x}$ is the applied focusing force proportional to the displacement $\mathbf{x}$, and $-\kappa_2(t)\cdot\mathbf{v}$ is the focusing force proportional to the average flow velocity $\mathbf{v}$. Here, q and m are the particle charge and mass, respectively, $n(\mathbf{x},t)$ is the particle number density, $\mathbf{v}(\mathbf{x},t)$ is the average flow velocity, and

$\varphi(\mathbf{x},t)$ is the space-charge potential generated by the charged particles. The conducting boundaries are assumed to be far away, i.e., $|\mathbf{x}_w| \to \infty$. The time-dependent tensors $\kappa_1(t)$ and $\kappa_2(t)$ include all of the known linear focusing forces as special cases. In a Paul trap, the focusing coefficients are

$$\kappa_1(t) = Diag[\kappa_{xx}(t), \kappa_{yy}(t), \kappa_{zz}(t)], \quad \kappa_2(t) = 0\,. \tag{5}$$

Here, $Diag[a,b,c]$ denotes the 3×3 diagonal matrix with diagonal components a, b, and c. For a standard Paul trap, the transverse field is a quadrupole potential, and $\kappa_{xx}(t) = -\kappa_{yy}(t)$ are the (oscillatory) transverse focusing coefficients, and the focusing coefficient $\kappa_{zz}(t) > 0$ provides longitudinal confinement of the particles in the z-direction. [In Eq. (5), we do not generally require that $\kappa_{xx}(t) = -\kappa_{yy}(t)$. The meaning of Paul trap here is a bit more general than the standard convention.] In a Penning trap, transverse confinement is provided by a uniform axial magnetic field $B_0(t)\,\mathbf{e}_z$, and $\kappa_1(t)$ and $\kappa_2(t)$ are given by

$$\kappa_1(t) = \begin{pmatrix} -\frac{1}{2}\omega_z^2(t) & -\Omega'(t) & 0 \\ \Omega'(t) & -\frac{1}{2}\omega_z^2(t) & 0 \\ 0 & 0 & \omega_z^2(t) \end{pmatrix},$$
$$\kappa_2(t) = \begin{pmatrix} 0 & -2\Omega(t) & 0 \\ 2\Omega(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

where $\Omega(t) = qB_0(t)/2mc$ is the Larmor frequency, and $\omega_z^2(t) > 0$ is the focusing coefficient in the longitudinal direction. The term $\Omega'(t)$ denotes the time derivative of $\Omega(t)$, representing the force due to the inductive electric field when $B_0(t)$ varies with time. As a simple model, the pressure $P(\mathbf{x},t)$ is taken to be a scalar, and the energy balance equation for the fluid is assumed to have the polytropic form in Eq. (3), where γ is the polytropic index. We emphasize that the energy equation adopted in the present analysis is a simple theoretical model to allow analytical progress, as discussed in detail by Dubin[14] and Amiranashvili and Stenflo.[21] The physics conclusions obtained in this paper are not sensitive to this choice of model.

The class of nonlinear collective dynamical solutions of the one-component nonneutral plasma is specified by the following solution structures admitted by the system of fluid-Poisson equations (1)–(4). The density $n(\mathbf{x},t)$ is taken to be uniform inside an ellipsoid and zero outside. The shape and orientation of the ellipsoid depend on time t, and are determined from (see Fig. 1)

$$S(\mathbf{x},t) = D_{ij}(t)x_i x_j < 1\,. \tag{7}$$

Inside the ellipsoid, the field quantities are assumed to be of the form

$$n(\mathbf{x},t) = n(t), \quad v_i(\mathbf{x},t) = v_{ij}(t)x_j, \quad P(\mathbf{x},t) = p_0(t) - p_{ij}(t)x_i x_j\,. \tag{8}$$

Here, x_i $(i = 1,2,3)$ denotes the three configuration coordinates of the displacement vector $\mathbf{x}$, and v_i $(i = 1,2,3)$ denotes

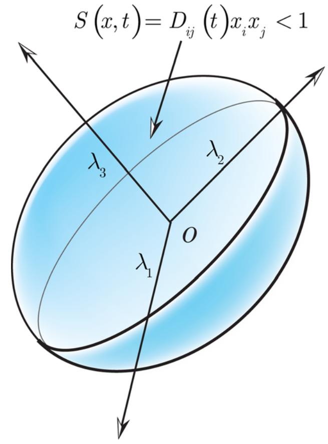

FIG. 1. (Color online) The density $n(\mathbf{x},t)$ is uniform inside the ellipsoid and zero outside. The shape and orientation of the ellipsoid depend on time, and are determined from $S(\mathbf{x},t) = D_{ij}(t)x_i x_j < 1$.

the three components of the flow velocity vector $\mathbf{v}$. There is an implicit summation over the repeated indices in Eq. (8). Equations (7) and (8) specify a particular space-time structure of the dynamical solutions. The velocity vector v_i is a linear function of the displacement vector x_i, and the coefficient is a time-dependent tensor $v_{ij}(t)$. The pressure P is given by a time-dependent function $p_0(t)$ plus a quadratic function of the displacement vector, specified by the symmetric tensor $p_{ij}(t)$.

The ellipsoid $S(\mathbf{x},t) = D_{ij}(t)x_i x_j < 1$ is determined self-consistently by the velocity field according to

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{v}\cdot\frac{\partial S}{\partial \mathbf{x}} = 0\,, \tag{9}$$

In terms of the matrices v and D, Eq. (9) can be expressed as

$$D' + v^T D + Dv = 0, \quad \text{or } D'_{ij} + v_{li}D_{lj} + D_{il}v_{lj} = 0, \tag{10}$$

where v^{T} denotes the transpose of v. If D_{ij} is initially symmetric and positive-definite, then the solution for D_{ij}(t) determined by Eq. (10) is symmetric and positive-definite at all subsequent times. This is because $v_{\mathrm{lj}}D_{\mathrm{lj}} + D_{\mathrm{lj}}v_{\mathrm{lj}}$ is symmetric, and D_{ij} cannot cross the boundary $|D_{ij}| = 0$, which corresponds to infinitely large density and pressure. In addition, if the initial conditions are chosen such that the pressure $P(\mathbf{x},t)$ vanishes at the boundary of the ellipsoid $S(\mathbf{x},t) = 1$ at $t = 0$, then Eqs. (3) and (9) guarantee that the pressure vanishes at the boundary at all time, i.e., $P(\mathbf{x},t)|_{S(\mathbf{x},t)} = 1$ for $t \geq 0$.

For given D_{ij} and total number of charged particles N, the solution to Poisson's equation (4) with boundary condition of $\varphi(|x| \to \infty) = 0$ is given by[9]

$$\varphi = -\frac{3Nq}{4}\int_0^\infty \frac{ds}{\sqrt{(\lambda_1^2+s)(\lambda_2^2+s)(\lambda_3^2+s)}} \times\left(1 - \frac{X^2}{\lambda_1^2+s} - \frac{Y^2}{\lambda_2^2+s} - \frac{Z^2}{\lambda_3^2+s}\right). \tag{11}$$

Here, λ_1^{-2}, λ_2^{-2}, λ_3^{-2} are the three eigenvalues of D_{ij}, and $(X,Y,Z)^T = Q^{-1}(x,y,z)^T$ denotes rotated coordinates. The orthogonal matrix Q is constructed from the three eigenvectors α_1, α_2, α_3 of D_{ij} as $Q=(\alpha_1, \alpha_2, \alpha_3)$. Note that λ_i and Q are uniquely determined by the matrix D_{ij}.

The self-similar solution has the form of homogeneous deformations, and the solution ansatz is similar to that constructed by Dubin[14] for a time-independent Penning trap. The fact that homogeneous deformations of flows lead to exact solutions of hydrodynamic equation system is well-known. Such solutions can be traced back to the 18th century and were first obtained for gravitating fluids by Chandrasekhar.[22]

Assuming the solution ansatz in Eq. (8), we find that the spatial dependence in the fluid equations (1)–(4) drops out, and the system reduce to a set of ordinary differential equations (ODEs) for the density $n(t)$, velocity matrix $v_{ij}(t)$, pressure matrix $p_{ij}(t)$, and $p_0(t)$ given by

$$n'(t) + nTr(v) = 0\,, \tag{12}$$

$$v' + vv - \frac{q}{m}QEQ^{-1} + \kappa_1 + \kappa_2 v - \frac{2p}{mn} = 0\,, \tag{13}$$

$$p' + v^T p + pv - \gamma p\frac{n'}{n} = 0\,, \tag{14}$$

$$\left(\frac{p_0}{n^\gamma}\right)' = 0\,. \tag{15}$$

Here, $Tr(v)$ denotes the trace of v, and E is the matrix representation of the self-electric field, which expresses the electric field in (X, Y, Z) coordinates as $E_{ij}X_j$. From Eq. (11), we obtain

$$E_{ij} = Diag(E_1, E_2, E_3)\,,$$

$$E_1 = \frac{3Nq}{2\lambda_1^3}G\left(\frac{\lambda_2}{\lambda_1},\frac{\lambda_3}{\lambda_1}\right), \quad E_2 = \frac{3Nq}{2\lambda_2^3}G\left(\frac{\lambda_1}{\lambda_2},\frac{\lambda_3}{\lambda_2}\right),$$
$$E_3 = \frac{3Nq}{2\lambda_3^3}G\left(\frac{\lambda_1}{\lambda_3},\frac{\lambda_2}{\lambda_3}\right),$$

$$G(u,v) \equiv \int_0^\infty \frac{ds}{\sqrt{(1+s)(u^2+s)(v^2+s)}}\,. \tag{16}$$

Note that the space-charge force $qQEQ^{-1}/m$ in Eq. (13) is uniquely determined by the matrix D_{ij}.

Equations (10) and (12)–(14) form a system of ODEs for D_{ij}, n, v_{ij}, and p_{ij}, which determines a class of nonlinear dynamical solutions of the one-component plasma in the external focusing field.

For certain forms of applied focusing fields, Eqs. (10) and (12)–(14) admit solutions with reduced dimensions. For example, in the time-dependent Paul trap described by Eq. (5), the system obviously admits solutions with $Q=I$, where I is the identity matrix, and diagonal solutions with v_{ij}, p_{ij}, and D_{ij},

$$v_{ij} = Diag[u_1, u_2, u_3], \quad p_{ij} = Diag[p_1, p_2, p_3],$$
$$D_{ij} = Diag[D_1, D_2, D_3]. \tag{17}$$

The ODE system for n, u_i, p_i, and D_i is then given by

$$n'(t) + n\sum_{i=1}^{3} u_i = 0\,, \tag{18}$$

$$u_i' + u_i^2 - \frac{q}{m}E_i + \kappa_{ii} - \frac{2p_i}{mn} = 0\,, \tag{19}$$

$$p_i' + 2u_i p_i - \gamma p_i\frac{n'}{n} = 0, \tag{20}$$

$$D_i' + 2u_i D_i = 0. \tag{21}$$

In the above equations, there is no summation over a repeated index i.

The dynamics in three dimensions are coupled through the space-charge potential and density. From Eq. (21), we obtain $u_i = \lambda_i'/\lambda_i$, where $\lambda_i = 1/\sqrt{D_i}$. Then Eq. (19) reduces to an equation for λ_i,

$$\lambda_i'' + \kappa_{ii}\lambda_i - \frac{q}{m}E_i\lambda_i - \frac{2p_i\lambda_i}{mn} = 0\,. \tag{22}$$

Equation (22) has a similar form to the familiar envelope equation for charged particle beams in a periodic focusing lattice. To see this, let us consider the special case where $\gamma=2$, $\kappa_{xx}=\kappa_{yy}=\kappa_r$, and the beam cross-section is circular, i.e., $\lambda_1=\lambda_2=r_b$. We further assume that the ellipsoid is very long, i.e., $\lambda_1=\lambda_2=r_b=\lambda_3$, and the focusing in the z-direction is sufficiently weak that the charge bunch is uniform in the z-direction over a scale-length comparable to r_b. Then $E_1 = E_2 = 3Nq/(\lambda_3 2r_b^2)$,[9] where $N = 4\pi n r_b^2\lambda_3/3 = const.$ is the total number of particles in the long charge bunch. From Eq. (18), we obtain $n = N_l/\pi r_b^2$, where $N_l \equiv 3N/4\lambda_3$ represents the line density. Equation (20) can be integrated to give $p_i\lambda_i^2 n^{-2} = const$. Making use of Eq. (15), we obtain the pressure solution as

$$P = p_0(t)\left(1 - \frac{x^2}{\lambda_1^2} - \frac{y^2}{\lambda_2^2} - \frac{z^2}{\lambda_3^2}\right), \tag{23}$$

which vanishes on the plasma boundary. The equations for rb = λ1 = λ2 then becomes

$$r_b'' + \kappa_r r_b - \frac{K_b}{r_b^2} - \frac{\varepsilon^2}{r_b^3} = 0\,. \tag{24}$$

Here, $K_b \equiv 2N_l q^2/m$ and $\varepsilon^2 \equiv 2p_r r_b^2 n^{-2}N_l/m$ represent the self-field perveance and the transverse emittance-squared. Note that the emittance is constructed from the constant of the motion $p_r r_b^2 n^{-2}$, and the line density N_l, which is approximately constant for a long charge bunch. The envelope equation (24) is identical to Eq. (6.60) in Ref. 9.

Another interesting example is the time-dependent Penning trap given by Eq. (6). It is a well-known fact that in a Penning trap, the single-particle transverse equations of motion transform to uncoupled linear oscillator equations in a frame rotating with the instantaneous Larmor frequency $\Omega(t)$.[26] The rotation matrix is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta = -\int_0^t \Omega(t)dt\,.$$

Here, we show that the macroscopic fluid dynamics also enjoys this desirable transformation property. The density n, velocity matrix v_{ij}, pressure matrix p_{ij}, and shape matrix D_{ij} in the laboratory frame are transformed to their counterparts $\bar{n}$, $\bar{v}_{ij}$, $\bar{p}_{ij}$, $\bar{D}_{ij}$ in the rotating frame as

$$n = \bar{n},\, D = \bar{D}R(\theta)\,,\, p = R(-\theta)\bar{p}R(\theta)\,, \tag{25}$$

$$v = R(-\theta)\bar{v}R(-\theta) + \Omega\frac{dR(\theta)}{d\theta}R(\theta)\,. \tag{26}$$

Substituting Eqs. (25) and (26) into Eqs. (1) and (4), we find that the $\bar{n}$, $\bar{v}_{ij}$, $\bar{p}_{ij}$, and $\bar{D}_{ij}$ satisfy the following equations in the rotating frame,

$$\bar{n}'(t) + \bar{n}Tr(\bar{v}) = 0\,, \tag{27}$$

$$\bar{v}' + \bar{v}\bar{v} - \frac{q}{m}\bar{Q}\bar{E}\bar{Q}^{-1} + \bar{\kappa}_1 - \frac{2\bar{p}}{m\bar{n}} = 0\,, \tag{28}$$

$$\bar{p}' + \bar{v}^T\bar{p} + \bar{p}\bar{v} - \gamma\bar{p}\frac{\bar{n}'}{\bar{n}} = 0\,, \tag{29}$$

$$\bar{D}' + \bar{v}^T\bar{D} + \bar{D}\bar{v} = 0. \tag{30}$$

Here, the transformed focusing matrix $\bar{\kappa}_1$ is diagonal,

$$\bar{\kappa}_1 = Diag\left[\Omega^2(t) - \frac{1}{2}\omega_z^2(t), \Omega^2(t) - \frac{1}{2}\omega_z^2(t), \omega_z^2(t)\right], \tag{31}$$

and there is no $\bar{\kappa}_2$ term in the rotating frame. This is similar in form to the case of a Paul trap in the laboratory frame. The difference is that the (1,1) and the (2,2) components of $\bar{\kappa}_{1ij}$ are the same, whereas in the case of a standard Paul trap, $\kappa xx = -\kappa yy$. Because of this, Eqs. (27)–(30) admit diagonal solutions of the form in Eq. (17) with $\bar{Q} = I$, $\bar{p}_1 = \bar{p}_2 = p_r$, and $\bar{D}_1 = \bar{D}_2 = 1/r_b^2$. Equations (27) and (29) can be integrated to give $p_r r_b^2/n^\gamma = const.$ and $p_z z_b^2/n^\gamma = const.$, or equivalently, $p_r r_b^{2+2\gamma} z_b^\gamma = const.$ and $p_z r_b^{2\gamma} z_b^{2+\gamma} = const.$, where using $n^\gamma r_b^2 z_b^\gamma = (3N/4\pi)^\gamma = const.$. The corresponding nonlinear envelope equations for rb and zb are given by

$$r_b'' + \left(\Omega^2 - \frac{1}{2}\omega_z^2(t)\right)r_b - \frac{3Nq}{2r_b^2}G\left(1, \frac{z_b}{r_b}\right) - \frac{\varepsilon_{r,\gamma}^2}{r_b^{2\gamma-1}z_b^{\gamma-1}} = 0\,, \tag{32}$$

$$z_b'' + \omega_z^2 z_b - \frac{3Nq}{2z_b^2}G\left(\frac{r_b}{z_b}, \frac{r_b}{z_b}\right) - \frac{\varepsilon_{z,\gamma}^2}{r_b^{2\gamma-2}z_b^\gamma} = 0\,, \tag{33}$$

where $\varepsilon_{r,\gamma}^2 \equiv 8\pi p_r r_b^{2\gamma+2} z_b^\gamma/3mN$ and $\varepsilon_{z,\gamma}^2 \equiv 8\pi p_z r_b^{2\gamma} z_b^{2+\gamma}/3mN$ are two constants of the motion. Here, $p_z = \bar{p}_3$ and $\bar{D}_3 = 1/z_b^2$. For a time-dependent Penning trap, Ω and ωz are time-dependent, and the nonlinear dynamical solutions are described by Eqs. (32) and (33). It can be shown that Eq. (32) reduces exactly to Eq. (24) when $\gamma = 2$ and the transverse focusing is weak and the plasma ellipsoid is long, i.e., $z_b \gg r_b$.

When Ω and ω_z are time-independent, Eqs. (27)–(30) possess another class of stationary ($\partial/\partial t = 0$) solutions with $\bar{Q} = I$, $\bar{D} = Diag[1/r_b^2, 1/r_b^2, 1/z_b^2] = const.$, $\bar{p} = Diag[p_r, p_r, p_z] = const$, and

$$v = \begin{pmatrix} 0 & \omega_r & 0 \\ -\omega_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = const \tag{34}$$

In this case, Eqs. (27)–(30) reduce to

$$\omega_r^2 + \frac{qE_r}{m} - \Omega^2 + \frac{1}{2}\omega_z^2(t) + \frac{2p_r}{mn} = 0\,, \tag{35}$$

$$\frac{qE_z}{m} - \omega_z^2 + \frac{2p_z}{mn} = 0\,. \tag{36}$$

When the nonneutral plasma ellipsoid is long in the z-direction and the transverse pressure p_r is negligibly small, Eq. (35) recovers to the well-known equilibrium radial force-balance equation for a cold nonneutral plasma column in a Penning trap (in the un-rotated laboratory frame)[23] as a special case with $qE_r/m = 3Nq^2/2z_b r_b^2 m = \omega_p^2/2$. For other types of linear focusing devices, such as rotating-radio-frequency traps and the Mobius accelerator, the nonlinear dynamical solutions described by Eqs. (8)–(14) can be calculated in a straightforward manner, using the particular κ_1 and κ_2 for each trap. Because of page limitations, these applications will be presented in a future paper.

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[1]W. Paul and H. Steinwede, Zeitschrift fr Naturforschung **8A**, 448 (1953).
[2]J. Bollinger, D. Heinzen, F. Moore, W. Itano, and D. J. Wineland, Phys. Rev. A **48**, 525 (1993).
[3]F. M. Penning, Physica (Utrecht) **3**, 873 (1936).
[4]J. Malmberg and T. O'Neil, Phys. Rev. Lett. **39**, 1333 (1977).
[5]T. O'Neil, Phys. Fluids **23**, 2216 (1980).
[6]J. Bollinger and D. Wineland, Phys. Rev. Lett. **53**, 348 (1984).
[7]C. Driscoll, J. Malmberg, and K. Fine, Phys. Rev. Lett. **60**, 1290 (1988).
[8]J. Fajans and L. Friedland, Am. J. Phys. **69**, 1096 (2001).
[9]R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (Imperial College Press and World Scientific, London, 2001).
[10]T. Hasegawa, M. Jensen, and J. Bollinger, Phys. Rev. A **71**, 023406 (2005).
[11]R. Talman, Phys. Rev. Lett. **74**, 1590 (1995).
[12]L. Turner, Phys. Fluids **30**, 3196 (1987).
[13]D. Dubin, Phys. Rev. Lett. **66**, 2076 (1991).
[14]D. Dubin, Phys. Fluids B **5**, 295 (1993).
[15]L. Turner and J. M. Finn, Phys. Plasmas **2**, 1378 (1995).
[16]D. Dubin and T. O'Neil, Rev. Mod. Phys. **71**, 87 (1999).
[17]D. Dubin, Phys. Plasmas **12**, 042107 (2005).
[18]N. D. Naumov, Plasma Phys. Rep. **30**, 269 (2004).
[19]A. R. Karimov, L. Stenflo, and M. Y. Yu, Phys. Plasmas **16**, 102303 (2009).
[20]A. Olivetti, J. Barré, B. Marcos, F. Bouchet, and R. Kaiser, Phys. Rev. Lett. **103**, 1 (2009).
[21]S. Amiranashvili and L. Stenflo, Phys. Rev. E **67** (2003).
[22]S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale University Press, New Haven, CT, 1968).
[23]R. C. Davidson, *Physics of Nonneutral Plasmas*, 2nd ed. (Imperial College Press and World Scientific, London, 2001), p. 41.