Generalized Courant–Snyder theory and Kapchinskij–Vladimirskij distribution for high-intensity beams in a coupled transverse focusing lattice[a)]

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The Courant–Snyder (CS) theory and the Kapchinskij–Vladimirskij (KV) distribution for high-intensity beams in an uncoupled focusing lattice are generalized to the case of coupled transverse dynamics. The envelope function is generalized to an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are noncommutative. In an uncoupled lattice, the KV distribution function, first analyzed in 1959, is the only known exact solution of the nonlinear Vlasov-Maxwell equations for high-intensity beams including self-fields in a self-consistent manner. The KV solution is generalized to high-intensity beams in a coupled transverse lattice using the generalized CS invariant. This solution projects to a rotating, pulsating elliptical beam in transverse configuration space. The fully self-consistent solution reduces the nonlinear Vlasov-Maxwell equations to a nonlinear matrix ordinary differential equation for the envelope matrix, which determines the geometry of the pulsating and rotating beam ellipse. These results provide us with a new theoretical tool to investigate the dynamics of high-intensity beams in a coupled transverse lattice. A strongly coupled lattice, a so-called N-rolling lattice, is studied as an example. It is found that strong coupling does not deteriorate the beam quality. Instead, the coupling induces beam rotation and reduces beam pulsation. © *2011 American Institute of Physics*. [doi:10.1063/1.3574919]

I. INTRODUCTION

Modern high-intensity beams have many important applications including high energy density physics, ion-beam-driven fusion, accelerator-driven subcritical systems, high-flux neutron sources, and light sources. Because it is critical to increase beam intensities as much as possible for these applications, it is becoming increasingly important to understand the self-field effects of high-intensity beams including both self-electric and self-magnetic fields in a fully self-consistent manner, from the nonlinear Vlasov-Maxwell equations.[1] In an uncoupled lattice, the Kapchinskij–Vladimirskij (KV) distribution function analyzed in 1959 (Ref. 2) is the only known exact self-consistent solution of the nonlinear Vlasov-Maxwell equations for high-intensity beams, and it provides us with the basic theoretical understanding of high-intensity beam dynamics in an uncoupled lattice. In practical accelerators and beam transport systems, the transverse coupling between the horizontal and vertical directions, induced by error fields and misalignment, is always a significant effect.[3–8] Strong coupling of the transverse dynamics is introduced intentionally in certain type of cooling channels and in the final focusing system for high energy density physics experiment,[9] as well as in the conceptual design of the Mbius accelerator.[10] A beam transport system with strong coupling was implemented in the spiral line induction accelerator (SLIA),[11–17] which reached up to 10 kA electron current at 5 MeV beam energy. The success of the proof of concept experiment at SLIA demonstrated the potential advantages of using a strongly coupled lattice for transporting high-intensity beams.

In this paper, we present a theoretical framework to study high-intensity beam dynamics in a coupled lattice using the Vlasov-Maxwell equations. We generalize the classical KV solution and the associated nonlinear envelope equations for high-intensity beams to the case of a coupled lattice.[18] To construct the generalized KV solution for high-intensity beams in a coupled lattice, we need to first generalize the Courant–Snyder (CS) theory for a single charged particle to the case of a couple lattice.[19–21] In particular, it is necessary to find a generalized CS invariant. It turns out that the original CS theory for one degree of freedom can be elegantly generalized to arbitrary degree of freedom using a time-dependent symplectic transformation technique. The generalized CS theory gives a complete description of the coupled transverse dynamics and has the same structure as the original CS theory for one degree of freedom. The four basic components of the original CS theory that have physical importance, i.e., the envelope equation, phase advance, transfer matrix, and the CS invariant, all have their counterparts, with remarkably similar expressions, in the generalized CS theory developed here. The unique feature of the generalized CS theory presented here is the non-Abelian (noncommutative) nature of the theory. In the generalized theory, the envelope function is generalized to an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are not commutative. The generalized theory gives a parametrization of the 4D symplectic transfer matrix [Eq. (56)] that has the same structure as the parametrization of the 2D symplectic transfer matrix [Eq. (21)] in the original CS theory.

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Using the generalized CS invariant, we can generalize the KV solution to the case of a coupled lattice. In the coupled case, the generalized KV distribution that solves the nonlinear Vlasov-Maxwell system projects to a rotating, pulsating beam with elliptical cross-section in transverse configuration space with constant density inside the beam. Both the major and minor radii and the tilting angle of the elliptical cross-section are functions of the time [see Fig. 3(b)], in contrast with the pulsating upright elliptical beam cross-section for the uncoupled case [see Fig. 3(a)]. We apply the theoretical results to the case of a strongly coupled lattice called the N-rolling lattice, which consists of N equally spaced quadrupole magnets, each of which rotates by an angle of π/N relative to its predecessor. It is found that strong coupling does not deteriorate the beam quality. Instead, the coupling induces beam rotation and reduces the beam pulsation. The oscillation amplitude of the transverse dimensions decreases for increasing number of magnets, and the beam cross-section is asymptotic to a rigid rotor profile as the number of magnet increases.

The paper is organized as follows. In Sec. II, the Vlasov-Maxwell system of equations for high-intensity beams in a coupled focusing lattice is presented, and in Sec. III we review the classical CS theory and KV solution for high-intensity beams in uncoupled lattices. Then the CS theory is generalized to coupled lattices in Sec. IV using a time-dependent canonical transformation technique. The generalized KV solution and examples of the N-rolling lattices are given in Sec. V.

II. VLASOV-MAXWELL SYSTEM FOR HIGH-INTENSITY BEAM IN COUPLED FOCUSING LATTICE

In a coupled transverse focusing lattice, the Vlasov-Maxwell equations that govern the evolution of the distribution function f of a high-intensity beam and the corresponding normalized space-charge potential ψ are

$$\frac{\partial f}{\partial s} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - (\nabla\psi + \kappa_{\mathbf{x}}\mathbf{x} + \kappa_{\mathbf{v}}\mathbf{v})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0\,, \tag{1}$$

$$\nabla^2\psi = \frac{-2\pi K_b}{N_b}\int f dv_x dv_y\,. \tag{2}$$

Here, particle motion in the beam frame is assumed to be nonrelativistic, ψ is the space-charge potential normalized by $\gamma_b^3 m\beta_b^2 c^2/q_b$, $\beta_b c$ is the directed beam velocity in the longitudinal direction, $\gamma_b = \left(1-\beta_b^2\right)^{-1/2}$ is the relativistic mass factor, $s=\beta_b ct$ is the time variable normalized by $1/\beta_b c$, $K_b = 2N_b q_b^2/\gamma_b^3 m\beta_b^2 c^2$ is the beam self-field perveance, $N_b = \int f dx dy dv_x dv_y$ is the line density, $\mathbf{x}=(x,y)^T$ represents the transverse displacement of a beam particle, and $\mathbf{v}=d\mathbf{x}/ds=\left(v_x,v_y\right)^T$ is the normalized transverse velocity. The corresponding normalization factor for the velocity is $\beta_b c$. In Eq. (1), the term $\kappa_{\mathbf{x}}\mathbf{x}$ is the linear focusing force proportional to the transverse displacement from the beam axis, and $\kappa_{\mathbf{v}}\mathbf{v}$ is that proportional to the transverse velocity. In a coupled lattice with quadrupole, skew-quadrupole, and solenoidal fields,

$$\kappa_{\mathbf{x}} = \kappa_q + \kappa_s,\;\; \kappa_q = \begin{pmatrix} \kappa_{qx} & \kappa_{qxy} \\ \kappa_{qyx} & \kappa_{qy} \end{pmatrix}, \tag{3}$$

$$\kappa_s = \begin{pmatrix} 0 & \Omega' \\ -\Omega' & 0 \end{pmatrix},\;\; \kappa_{\mathbf{v}} = \begin{pmatrix} 0 & 2\Omega \\ -2\Omega & 0 \end{pmatrix}, \tag{4}$$

where the matrix κ_q describes the focusing fields due to the quadrupole and skew-quadrupole magnets, and κ_s and $\kappa_{\mathbf{v}}$ describe the focusing field due to the solenoidal lattice. First, let us consider κ_q, whose components can be expressed as

$$\kappa_{qx} = -\kappa_{qy} = \kappa_{q0}(s)\cos 2\tau(s), \tag{5}$$

$$\kappa_{qxy} = \kappa_{qyx} = \kappa_{q0}(s)\sin 2\tau(s). \tag{6}$$

Here, $\tau(s)$ is the rotation angle of the quadrupole magnet and $\kappa_{q0}(s)$ is the magnetic field gradient on axis normalized by $\gamma_b m\beta_b c^2/q$. In a standard FODO lattice,

$$\tau(s) = \begin{cases} 0, & -\dfrac{\eta}{4} < \dfrac{s}{S} < \dfrac{\eta}{4}\,, \\ \dfrac{\pi}{2}, & -\dfrac{\eta}{4} < \dfrac{s}{S} - \dfrac{1}{2} < \dfrac{\eta}{4}\,, \\ \pi, & -\dfrac{\eta}{4} < \dfrac{s}{S} - 1 < \dfrac{\eta}{4}, \end{cases} \tag{7}$$

$$\kappa_{q0}(s) = \begin{cases} \dfrac{qB\prime_q}{\gamma_b m\beta_b c^2}, & \dfrac{s}{S} \in \cup_{i=0}^{2}\left[\dfrac{i}{2} - \dfrac{\eta}{4}, \dfrac{i}{2} + \dfrac{\eta}{4}\right], \\ 0, & \text{otherwise}\,, \end{cases} \tag{8}$$

where η is the filling factor of the FODO lattice, S is the lattice period, and B'_q is the field gradient on axis for the quadrupole magnets. Equation (7) gives $\kappa_{qxy} = \kappa_{qyx} = 0$, implying that there is no coupling between x and y due to the quadrupole magnets. If a misalignment of a FODO quadrupole magnet occurs, $\tau(s)$ will be different from its design value of π or $\pi/2$, resulting in a nonvanishing $\kappa_{qxy} = \kappa_{qyx}$. It is not difficult to design a quadrupole lattice which strongly couples the x and y dynamics. For example, we can choose the following N-rolling quadrupole lattice which consists of N equally spaced quadrupole magnets (see Fig. 1). Each magnet rotates by an angle of $\delta = \pi/N$ relative to its predecessor, and the filing factor for each magnet is η/N,

$$\tau(s) = \begin{cases} 0, & -\dfrac{\eta}{2N} < \dfrac{s}{S} < \dfrac{\eta}{2N}\,, \\ \dfrac{\pi}{N} & -\dfrac{\eta}{2N} < \dfrac{s}{S} - \dfrac{1}{N} < \dfrac{\eta}{2N}\,, \\ \dfrac{2\pi}{N} & -\dfrac{\eta}{2N} < \dfrac{s}{S} - \dfrac{2}{N} < \dfrac{\eta}{2N}\,, \\ \cdots & \\ \pi, & -\dfrac{\eta}{2N} < \dfrac{s}{S} - 1 < \dfrac{\eta}{2N}\,, \end{cases} \tag{9}$$

$$\kappa_{q0}(s) = \begin{cases} \dfrac{qB'_q}{\gamma_b m\beta_b c^2}, & \frac{s}{S} \in \cup_{i=0}^{N}\left[\dfrac{i}{N} - \dfrac{\eta}{2N}, \dfrac{i}{N} + \dfrac{\eta}{2N}\right], \\ 0, & \text{otherwise}\,. \end{cases} \tag{10}$$

This example will be studied in Sec. V. The standard FODO lattice is a special case of the N-rolling lattice with $N=2$.

The solenoidal field in Eq. (4) is expressed in terms of the normalized Larmor frequency $\Omega(s) = qB_z(s)/2\gamma_b\beta_b mc^2$. When the solenoidal component exists, the $(\mathbf{x},\mathbf{v})$ coordinates in Eqs. (1) and (2) are not canonical coordinates, i.e., their dynamics

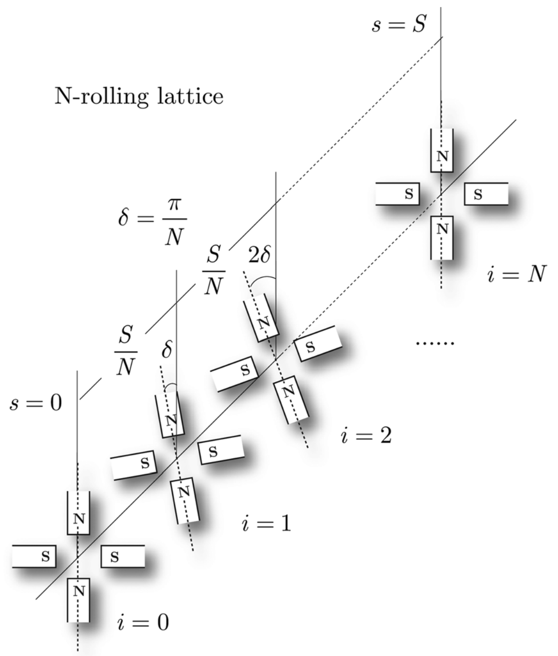

FIG. 1. An N-rolling lattice consists of N equally spaced quadrupole magnets. Each magnet rotates by an angle of $\delta = \pi/N$ relative to its predecessor. The standard FODO lattice is a special case of the N-rolling lattice with $N = 2$.

cannot be cast into the canonical Hamiltonina form. To determine the dynamics of a single particle or the characteristics of the Vlasov equation (1), it is advantageous to use a canonical Hamiltonina form for the dynamical equations. This can be achieved by introducing the canonical momentum defined as

$$p_x = v_x - \Omega y\,, \tag{11}$$

$$p_y = v_y + \Omega x\,. \tag{12}$$

The corresponding Hamiltonian is

$$H_c = z^\dagger A_c z + \psi\,,\; z = \left(x, y, p_x, p_y\right)^\dagger, \tag{13}$$

$$A_c = \begin{pmatrix} \kappa & R \\ R^\dagger & \frac{I}{2} \end{pmatrix}, \kappa = \begin{pmatrix} \frac{\Omega^2}{2} + \frac{\kappa_{qx}}{2} & \kappa_{qxy} \\ \kappa_{qyx} & \frac{\Omega^2}{2} + \frac{\kappa_{qy}}{2} \end{pmatrix}, R = \begin{pmatrix} 0 & -\frac{\Omega}{2} \\ \frac{\Omega}{2} & 0 \end{pmatrix}. \tag{14}$$

The Vlasov-Maxwell equations in the canonical coordinates $(\mathbf{x}, \mathbf{p})$ are

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \tag{15}$$

$$\nabla^2 \psi = \frac{-2\pi K_b}{N_b} \int f dp_x dp_y\,. \tag{16}$$

The momentum dependent part of the focusing force, i.e., the term proportional to R, can be transformed away if we transform to the local Larmor frame.[1,19] For simplicity of presentation, we will assume that there is no solenoidal field and consider here only the coupling due to skew-quadrupole magnets given by Eq. (3) in this paper.

The $-\nabla\psi$ term in Eq. (1) describes the self-field force and is nonlinearly coupled to f through Eq. (2). Equations (1) and (2) form a set of nonlinear integro-differential equations, whose analytical solutions are difficult to find in general.

III. COURANT–SNYDER THEORY AND KAPCHINSKIJ–VLADIMIRSKIJ DISTRIBUTION FOR HIGH-INTENSITY BEAM IN UNCOUPLED FOCUSING LATTICE

For the case of an uncoupled lattice, i.e., $\kappa_{qxy} = \kappa_{qyx} = 0$, Eqs. (1) and (2) admit a remarkable exact solution known as the KV distribution,[2] which has played a crucial role in high-intensity beam physics.[22–25] The KV distribution function is constructed as a function of the CS invariants of the transverse dynamics.[26] Therefore, we first review the CS theory for a uncoupled lattice.

The transverse dynamics of a charged particle in a linear focusing lattice $\kappa_\xi(s)$ is described by an oscillator equation with time-dependent frequency

$$\ddot{\xi} + \kappa_\xi(s)\xi = 0\,, \tag{17}$$

where ξ represents one of the transverse coordinates, either x or y, and $\ddot{\xi}$ represents $d^2\xi/dt^2$. For a quadrupole lattice, $\kappa_x(s) = -\kappa_y(s)$. The CS theory[26] gives a complete description of the solution to Eq. (17) and serves as the fundamental theory that underlies the design of modern accelerators and storage rings. There are four main components of the CS theory: the envelope equation, the phase advance, the transfer matrix, and the CS invariant. The Courant–Snyder theory can be summarized as follows. Because Eq. (17) is linear, its solution can be expressed as a time-dependent linear map from the initial conditions, i.e., $(\xi, \dot{\xi})^\dagger = M(s)(\xi_0, \dot{\xi}_0)^\dagger$, where

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\phi + \alpha_0 \sin\phi] & \sqrt{\beta\beta_0}\sin\phi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\phi + \frac{\alpha_0 - \alpha}{\sqrt{\beta\beta_0}}\cos\phi & \sqrt{\frac{\beta_0}{\beta}}[\cos\phi - \alpha\sin\phi] \end{pmatrix} \tag{18}$$

with $\xi_0 = \xi(s=0)$, $\dot{\xi}_0 = \dot{\xi}(s=0)$, $\beta_0 = \beta(s=0)$, and $\alpha_0 = \alpha(s=0)$. The superscript $\dagger$ denotes the transpose operation. The time-dependent functions $\alpha(s)$, $\beta(s)$, and $\phi(s)$ in the transfer matrix $M(s)$ are directly related to the envelope function $w(s)$ by

$$\beta(s) = w^2(s)\,,\; \alpha(s) = -w\dot{w}\,,\; \phi(s) = \int_0^s \frac{ds}{\beta(s)}\,. \tag{19}$$

The envelope function $w(s)$ satisfies the nonlinear envelope equation

$$\ddot{w} + \kappa_\xi(s) w = w^{-3}\,. \tag{20}$$

The physical meanings of β^{-1} and ϕ correspond to the phase advance rate and the phase advance, respectively. The transfer matrix $M(s)$ is symplectic and has the following decomposition:[27]

$$M(s) = \begin{pmatrix} w & 0 \\ \dot{w} & \frac{1}{w} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}. \quad (21)$$

The well-known CS invariant[26,28] is

$$I = \frac{\xi^2}{w^2} + \left(w\dot{\xi} - \dot{w}\xi\right)^2. \quad (22)$$

We emphasize that the CS theory provides an important mathematical scheme to parametrize the symplectic transfer matrix. The parameters of envelope, phase advance, and CS invariant furnished by the CS theory are of vital importance for beam physics. These parameters describe the physical dimensions and the emittance of the beam and set the foundation for many important concepts in beam physics, as is demonstrated in the construction of the KV distribution function for beams with strong space-charge field.

The oscillation dynamics with time-dependent frequency described by Eq. (17) is generic physics problem of great importance. It can be viewed as the second simplest physics problem. The associated envelope Equation (20) and the CS invariant (22) are manifestation of the underpinning symmetry of the dynamics.[28,29]

Because the CS invariants are valid for linear, uncoupled transverse force components, the KV distribution must self-consistently generates a linear, uncoupled space-charge force. The KV distribution indeed satisfies this requirement. It is given by[1,2]

$$f_{KV} = \frac{N_b}{\pi^2 \varepsilon_x \varepsilon_y} \delta\left(\frac{I_x}{\varepsilon_x} + \frac{I_y}{\varepsilon_y} - 1\right), \quad (23)$$

$$I_x = \frac{x^2}{w_x^2} + (w_x v_x - x\dot{w}_x)^2, \; I_y = \frac{y^2}{w_y^2} + \left(w_y v_y - y\dot{w}_y\right)^2. \quad (24)$$

Here, I_x and I_y are the CS invariants for the x- and y-motions, respectively, ε_x and ε_y are the constant transverse emittances, and w_x and w_y are the envelope functions satisfying the envelope equations,

$$\ddot{w}_x + \kappa_x w_x = w_x^{-3}, \; \ddot{w}_y + \kappa_y w_y = w_y^{-3}, \quad (25)$$

$$\kappa_x = \kappa_{qx} - \frac{2K_b}{a(a+b)}, \; \kappa_y = \kappa_{qy} - \frac{2K_b}{b(a+b)}, \quad (26)$$

$$a \equiv \sqrt{\varepsilon_x} w_x, \; b \equiv \sqrt{\varepsilon_y} w_y. \quad (27)$$

The density profile in the transverse configuration space projected by the distribution function f_{KV} in Eq. (23) is given by

$$n(x,y,s) = \int dv_x dv_y f_{KV} = \begin{cases} \frac{N_b}{\pi ab} = \text{const.}, & 0 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, \\ 0, & 1 < \frac{x^2}{a^2} + \frac{y^2}{b^2}, \end{cases} \quad (28)$$

which corresponds to a constant-density beam with elliptical cross-section and pulsating transverse dimensions a and b [see Fig. 3(a)]. The associated space-charge potential inside the beam, determined from Eq. (2), is given by

$$\psi = \frac{-K_b}{a+b}\left(\frac{x^2}{a} + \frac{y^2}{b}\right), \; 0 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1. \quad (29)$$

The KV distribution (23) reduces the original nonlinear Vlasov-Maxwell equations (1) and (2) to the two envelope equations in Eq. (25) for w_x and w_y or equivalently for $a = \sqrt{\varepsilon_x} w_x$ and $b = \sqrt{\varepsilon_y} w_y$ [Eq. (27)]. As the only known solution of the nonlinear Vlasov-Maxwell equations (1) and (2), the KV distribution and the associated envelope equations provide very important elementary theoretical tools for our understanding of high-intensity beam dynamics.[22–25] The KV distribution in Eq. (23) is constructed from the exact dynamical invariants I_x and I_y in Eq. (24), and constitutes an exact solution of the Vlasov equation (1), which also generates the uncoupled linear space-charge force assumed *a priori*.

In Sec. IV, we will first generalize the CS theory to the case of a coupled lattice, and then in Sec. V the KV solution will be generalized to high-intensity beams in a coupled focusing lattice, using the generalized CS invariant in Sec. IV.

IV. GENERALIZED COURANT–SNYDER THEORY FOR HIGH-INTENSITY BEAM IN A COUPLED FOCUSING LATTICE

We will generate the CS theory to coupled focusing lattice using a time-dependent canonical transformation technique.[30] Let us consider a linear, time-dependent Hamiltonian system with n-degree of freedom given by

$$\begin{aligned} H &= \frac{1}{2} z A(s) z^T, \\ z &= (x_1, x_2, ..., x_n, p_1, p_2, ..., p_n). \end{aligned} \quad (30)$$

Here, $A(s)$ is a $2n \times 2n$ time-dependent, symmetric matrix, and s is the time variable. The Hamiltonian in Eq. (13) has this form with $n = 2$. We introduce a time-dependent linear canonical transformation[30]

$$\bar{z} = S(s) z, \quad (31)$$

such that in the new coordinate $\bar{z}$, the transformed Hamiltonian has the form

$$\bar{H} = \frac{1}{2} \bar{z} \bar{A}(s) \bar{z}^T, \quad (32)$$

where $\bar{A}(s)$ is a targeted symmetric matrix. Because the transformation between z and $\bar{z}$ is required to be a canonical transformation, we have

$$\frac{\partial \bar{z}_j}{\partial z_k} J_{kl} \frac{\partial \bar{z}_j}{\partial z_l} = J_{ij}. \quad (33)$$

Here, J represents the $2n \times 2n$ unit symplectic matrix of order $2n$,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (34)$$

where I is the $n \times n$ unit matrix. From Eq. (31), Eq. (33) is equivalent to

$$SJS^T = J \,,$$

i.e., the matrix S specifying the coordination transformation between z and $\bar{z}$ needs to a symplectic matrix. In addition, the matrix $S(s)$ that renders this canonical transformation needs to satisfy a differential equation, which can be derived as follows. Hamilton's equation for z is

$$\dot{z} = J\nabla H \,. \tag{35}$$

With the quadratic form of the Hamiltonian in Eq. (30), Eq. (35) becomes

$$\begin{aligned}\dot{z}_j &= J_{ij}\frac{\partial H}{\partial z_j} = \frac{1}{2}J_{ij}\left(\delta_{lj}A_{lm}z_m + z_l A_{lk}\delta_{kj}\right)\\ &= \frac{1}{2}J_{ij}\left(A_{jm} + A_{mj}\right)z_m = J_{ij}A_{jm}z_m.\end{aligned} \tag{36}$$

In matrix notation without indices, it is expressed as

$$\dot{z} = JAz \,. \tag{37}$$

Because we require that in $\bar{z}$ the transformed Hamiltonian is given by Eq. (32), the following equation holds as well:

$$\dot{\bar{z}} = J\bar{A}\bar{z}. \tag{38}$$

Using Eq. (31), we can rewrite Eq. (38) as

$$\dot{\bar{z}} = J\bar{A}\bar{z} = J\bar{A}Sz \,. \tag{39}$$

Meanwhile, $\dot{\bar{z}}$ can be directly calculated from Eq. (31) by taking a time-derivative, which gives

$$\dot{\bar{z}} = \dot{S}z + S\dot{z} = \left(\dot{S} + SJA\right)z \,. \tag{40}$$

Combining Eqs. (39) and (40) gives the differential equation

$$\dot{S} = J\bar{A}S - SJA \,, \tag{41}$$

where S needs to satisfy if $\bar{z} = S(t)z$ is a canonical transformation.

The remarkable feature of Eq. (41) is that its solution S is always symplectic (i.e., $SJS^T = J$), if S is symplectic at $t = 0$. To prove this fact, we follow Leach[30] and consider the dynamics of the matrix $K = SJS^T$,

$$\begin{aligned}\dot{K} &= \dot{S}JS^T + SJ\dot{S}^T = 2\left[(J\bar{A}S - SJA)JS^T + SJ\left(-S\bar{A}J + AJS^T\right)\right]\\ &= 2\left[J\bar{A}SJS^T - SJS^T\bar{A}J\right] = 2[J\bar{A}K - K\bar{A}J].\end{aligned} \tag{42}$$

Equation (42) has a fixed point at $K = J$. If $S(s = 0)$ is symplectic, i.e., $K(s = 0) = J$, then $\dot{K} = 0$ and $K = J$ for all s, and S is symplectic for all s.

A more physical and geometric proof can be given from the viewpoint of the flow of S (see Fig. 2). Because A is symmetric, $JJ\bar{A} - \bar{A}^T JJ = 0$, which implies that $J\bar{A}$ belongs to the Lie algebra $sp(2n, R)$ of the Lie group of the symplectic matrix $Sp(2n, R)$. If $S \in Sp(2n, R)$ at a given t, then $J\bar{A}S$ belongs to the tangent space of $Sp(2n, R)$ at S, i.e., $J\bar{A}S \in T_S SP(2n, R)$. This is because if we examine the Lie group right action with

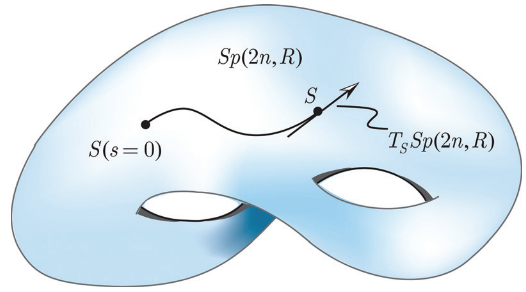

FIG. 2. (Color online) The space of the symplectic group $Sp(2n, R)$ and the flow of S on $Sp(2n, R)$. At any given time, $J\bar{A}S - SJA$ is always tangential to $Sp(2n, R)$. Therefore, the flow of S according to Eq. (41) is always on $Sp(2n, R)$.

$$S:\ a \mapsto aS \tag{43}$$

for any a in $Sp(2n, R)$ and the associated tangent map

$$T_S:\ T_a Sp(2n, R) \to T_{aS} Sp(2n, R), \tag{44}$$

it is evident that $J\bar{A}S$ is the image of the Lie algebra element $J\bar{A}$ under the tangential map T_S. This means that $J\bar{A}S$ is a "vector" tangential to the space of $Sp(2n, R)$ at S, if S is on $Sp(2n, R)$. The same argument applies to SJA as well. Consequently, the S dynamics will stay on the space of $Sp(2n, R)$ according to Eq. (41). Because the initial condition for S is arbitrary, we can always chose the initial condition such that S is symplectic at $s = 0$ and guarantee that the time-dependent canonical transformation satisfying Eq. (41) to be symplectic for all s.

We are now ready to develop the generalized Courant–Snyder theory for coupled transverse dynamics, using this technique of time-dependent canonical transformation. As indicated in Sec. II, for simplicity of presentation, we present here only the results for the case of the coupled dynamics induced by a skew-quadrupole component, i.e., $\kappa_{sq} \neq 0$, $R = 0$, and $\Omega = 0$. A treatment of the coupling due to the solenoidal lattice can be found in Ref. 31. We also assume the space-charge potential is quadratic in (x, y), i.e.,

$$-\nabla\psi = -\kappa_s \mathbf{x} \,,\ \kappa_s = \begin{pmatrix}\kappa_{sx} & \kappa_{sxy}\\ \kappa_{syx} & \kappa_{sy}\end{pmatrix}, \tag{45}$$

where $\kappa_{sxy} = \kappa_{syx}$. The Hamiltonian then takes the form of

$$H_c = z^\dagger A_c z \,, \tag{46}$$

$$A_c = \begin{pmatrix}\kappa & 0\\ 0 & \frac{I}{2}\end{pmatrix},\ \kappa = \kappa_q + \kappa_s. \tag{47}$$

Our objective is to solve the coupled system by finding the transfer matrix between the initial condition $z_0 = (x_0, y_0, \dot{x}_0, \dot{y}_0)^\dagger$ and $z = (x, y, \dot{x}, \dot{y})^\dagger$ at time s. We accomplish this goal by two time-dependent canonical transformations. The first step is to transform H_c into

$$\bar{H}_c = \bar{z}^\dagger \bar{A}_c \bar{z} \,,\ \bar{A}_c = \begin{pmatrix}\dfrac{\beta^{-1}}{2} & 0\\ 0 & \dfrac{\beta^{-1}}{2}\end{pmatrix}, \tag{48}$$

and the second step is to transform $\bar{H}_c$ into $\bar{\bar{H}}_c = 0$. Here, β is a time-dependent 2×2 matrix yet to be determined. As implied by its notation, the matrix of β is the generalized β function for the coupled dynamics. The physics that appears in the first step is the envelope matrix and the non-Abelian matrix envelope equation. The physics that appears in the second step is the phase advance. Let $\bar{z} = Sz$ be the transformation that transforms H_c into $\bar{H}_c$. From Eq. (41), the differential equation for S is

$$\dot{S} = 2(J\bar{A}_c S - SJA_c)\,. \tag{49}$$

The solution to Eq. (49) is

$$S = \begin{pmatrix} (w^{-1})^\dagger & 0 \\ -\dot{w} & w \end{pmatrix},$$

where w is the 2×2 envelope matrix satisfying the envelope matrix equation

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^\dagger w^{-1} \left(w^{-1}\right)^\dagger . \tag{50}$$

The β matrix in Eq. (48) is given by

$$\beta^{-1} = \left(w^{-1}\right)^\dagger w^{-1}, \tag{51}$$

which is remarkably similar to the phase advance rate $\beta^{-1} = 1/w^2$ in the original Courant–Snyder theory for one degree of freedom [see Eq. (19)]. The inverse transformation is

$$z = S^{-1}\bar{z}\,,\ S^{-1} = \begin{pmatrix} w^\dagger & 0 \\ w^{-1}\dot{w}w^\dagger & w^{-1} \end{pmatrix}. \tag{52}$$

The matrix S^{-1} is the non-Abelian generalization for the first matrix in the expression of the transfer matrix M in the original Courant–Snyder theory, i.e., the first term on the right hand side of Eq. (21).

The next step is to transform $\bar{H}_c$ into $\bar{\bar{H}}_c = 0$ with $\bar{\bar{A}}_c = 0$ by a transformation specified by $\bar{\bar{z}} = P\bar{z}$. Following the same procedure, the differential equation for P is

$$\dot{P} = P\dot{\phi}\,,\ \dot{\phi} \equiv \begin{pmatrix} 0 & -\left(w^{-1}\right)^\dagger w^{-1} \\ \left(w^{-1}\right)^\dagger w^{-1} & 0 \end{pmatrix}, \tag{53}$$

which admits solution of the form

$$P = \begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix}. \tag{54}$$

From the fact that P belongs to $Sp(4,R)$, we can readily show that $PP^\dagger = I$ and $Det(P) = 1$. Therefore, P corresponds to a rotation in the 4D phase space, $P \in SO(4)$. In this sense, $P^\dagger$ is the 4D non-Abelian generalization of the 2D rotation matrix in the expression of the transfer matrix M for the original Courant–Snyder theory, i.e., the second term on the right hand side of Eq. (21). Because $\dot{\phi}^\dagger = -\dot{\phi}$, it follows that $\dot{\phi}$ belongs to the Lie algebra $so(4)$ of the rotation group $SO(4)$, i.e., $\dot{\phi}$ is an infinitesimal generator of a 4D rotation. In other words, $\dot{\phi}$ is an "angular velocity" in 4D space, which is equivalent to a phase advance rate in 4D space. The 4D phase advance rate $\dot{\phi}$ is determined from the generalized phase advance matrix β^{-1} in Eq. (51).

Because $\bar{\bar{H}}_c = 0$, the dynamics of $\bar{\bar{z}}$ is trivial, i.e., $\bar{\bar{z}} = \bar{\bar{z}}_0$, and we have solved the Hamiltonian system H_c in $\bar{\bar{z}}$. From $\bar{\bar{z}} = PSz$ and $\bar{\bar{z}} = \bar{\bar{z}}_0$, we obtain the linear map between z_0 and z, i.e.,

$$z = S^{-1}P^{-1}\bar{\bar{z}} = S^{-1}P^{-1}\bar{\bar{z}}_0 = S^{-1}P^{-1}P_0S_0z_0\,. \tag{55}$$

Because $P \in SO(4,R)$, without loss of generality we select the initial condition $P_0 = P(t=0) = I$ to obtain $z = M_c z_0$,

$$\begin{aligned} M_c &= S^{-1}P^{-1}S_0 \\ &= \begin{pmatrix} w^\dagger & 0 \\ w^{-1}\dot{w}w^\dagger & w^{-1} \end{pmatrix} \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \begin{pmatrix} w_0^{-1\dagger} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}. \end{aligned} \tag{56}$$

The transfer matrix M_c in Eq. (56) is the 4D non-Abelian generalization of the transfer matrix in Eq. (21) for one degree of freedom. The similarities between M_c and M is evident from Eqs. (56) and (21). The generalized Courant–Snyder invariant for 4D coupled dynamics corresponding to the original Courant–Snyder invariant is

$$_{ICS} = \bar{\bar{z}}^\dagger \bar{\bar{z}} = z^\dagger S^\dagger P^\dagger PSz = z^\dagger S^\dagger Sz \tag{57}$$

$$= \mathbf{x}^\dagger w^{-1}w^{-1\dagger}\mathbf{x} + \left(\mathbf{v}^\dagger w^\dagger - \mathbf{x}^\dagger \dot{w}^\dagger\right)(w\mathbf{v} - \dot{w}\mathbf{x}), \tag{58}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ the phase advance has been removed due to the fact that P is a 4D rotation. In general, for any constant real positive definite matrix ϵ,

$$I_{CSg} = z^\dagger S^\dagger P^\dagger \epsilon PSz \tag{59}$$

is an invariant of the dynamics and should be called a generalized CS invariant as well. As pointed out in Ref. 6, for the coupled 4D transverse dynamics, there should be two (independent) invariants of this kind.

We now show that the generalized CS theory developed for coupled transverse dynamics recovers the original CS theory for dynamics with one degree of freedom as a special case. For the uncoupled transverse dynamics given by H_c with $\kappa_{xy} = \kappa_{yx} = 0$, κ is diagonal, and the matrix envelope equation (50) admits solutions with diagonal envelope matrix

$$w = \begin{pmatrix} w_x & 0 \\ 0 & w_y \end{pmatrix}.$$

Consequently, every matrix in Eq. (50) is diagonal, and the matrix operation is Abelian (commutative). The matrix envelope equation reduces to two decoupled envelope equations of the conventional form for w_x and w_y, i.e.,

$$\ddot{w}_x + w_x\kappa_x = w_x^{-3}\,, \tag{60}$$

$$\ddot{w}_y + w_y\kappa_y = w_y^{-3}\,. \tag{61}$$

The 2×2 matrix of phase advance rate β^{-1} reduces to a diagonal matrix as well, i.e.,

$$\beta^{-1} = \begin{pmatrix} w_x^{-2} & 0 \\ 0 & w_y^{-2} \end{pmatrix}.$$

The components of the phase advance (54) are

$$P_1 = \begin{pmatrix} \cos\phi_x & 0 \\ 0 & \cos\phi_y \end{pmatrix},$$
$$P_2 = \begin{pmatrix} \sin\phi_x & 0 \\ 0 & \sin\phi_y \end{pmatrix},$$

where $\dot{\phi}_x = w_x^{-2}$ and $\dot{\phi}_y = w_y^{-2}$ are the phase advances in the x- and y-directions. The transfer matrix reduces to

$$M = \begin{pmatrix} w_x & 0 & 0 & 0 \\ 0 & w_y & 0 & 0 \\ \dot{w}_x & 0 & w_x^{-1} & 0 \\ 0 & \dot{w}_y & 0 & w_y^{-1} \end{pmatrix} \times \begin{pmatrix} \cos\phi_x & 0 & -\sin\phi_x & 0 \\ 0 & \cos\phi_y & 0 & -\sin\phi_y \\ \sin\phi_x & 0 & \cos\phi_x & 0 \\ 0 & \sin\phi_y & 0 & \cos\phi_y \end{pmatrix} \times \begin{pmatrix} w_{x0}^{-1} & 0 & 0 & 0 \\ 0 & w_{y0}^{-1} & 0 & 0 \\ -\dot{w}_{x0} & 0 & w_{x0} & 0 \\ 0 & -\dot{w}_{y0} & 0 & w_{y0} \end{pmatrix}. \quad (62)$$

Apparently, the $(x,\dot{x})$ dynamics and the $(y,\dot{y})$ dynamics are decoupled, and the transfer matrices for $(x,\dot{x})$ and $(y,\dot{y})$ extracted from Eq. (62) are identical to that in Eq. (21) for one degree of freedom.

V. GENERALIZED KAPCHINSKIJ–VLADIMIRSKIJ DISTRIBUTION FOR A HIGH-INTENSITY BEAM IN COUPLED FOCUSING LATTICES

In this section, we generalize the classical KV solution described in Sec. III to the case of coupled transverse dynamics when $\kappa_{qxy} = \kappa_{qyx} \neq 0$, using the generalized CS invariant for coupled transverse lattice developed in Sec. IV. In the coupled case, the generalized KV distribution that solves the nonlinear Vlasov-Maxwell systems (1) and (2) projects to a rotating, pulsating beam with elliptical cross-section in transverse configuration space with constant density inside the beam. Both the dimensions a and b and the tilt angle θ are functions of $s = \beta_b ct$ [see Fig. 3(b)], in contrast with the pulsating upright elliptical beam cross-section for the uncoupled case [see Fig. 3(a)]. The rotating, pulsating beam with elliptical cross-section in transverse configuration space and constant density inside the beam, generates a coupled linear space-charge force in the form of Eq. (45), which allows us to apply the generalized CS invariant for the coupled transverse dynamics. The exact form of κ_s will be determined self-consistently [see Eq. (76)]. Our strategy is to use the generalized CS invariant to construct a

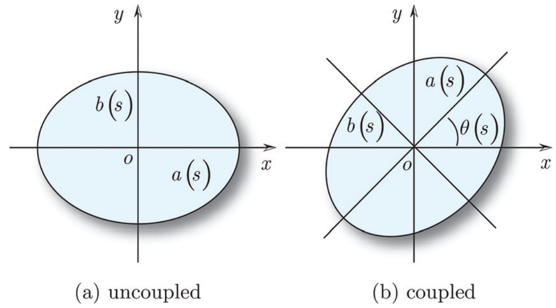

FIG. 3. (Color online) Beam cross-sections for the KV distribution. (a) Uncoupled lattice: the cross-section is determined from $0 \leq x^2/a^2 + y^2/b^2 < 1$; (b) Coupled lattice: the cross-section is determined from $0 \leq \mathbf{x}^T w^{-1} w^{-1T} \mathbf{x} < \varepsilon$.

generalized KV solution of the Vlasov equation (1), which also projects to a rotating, pulsating elliptical beam with constant density inside the beam. In this manner, a self-consistent solution of the nonlinear Vlasov-Maxwell equations (1) and (2) is found for high-intensity beams in a coupled transverse focusing lattice.

For clarity, we summarize the key results obtained in Sec. IV that will be used to construct the generalized KV solution. For a charged particle subject to the coupled linear focusing force and the coupled linear space-charge force

$$-\nabla\psi - \kappa_q \mathbf{x} = -\kappa \mathbf{x}, \ \ \kappa = \kappa_q + \kappa_s, \quad (63)$$

the generalized CS invariant is given by

$$I_{CS} = \mathbf{x}^\dagger w^{-1} w^{-1\dagger} \mathbf{x} + \left(\mathbf{v}^\dagger w^\dagger - \mathbf{x}^\dagger \dot{w}^\dagger\right)(w\mathbf{v} - \dot{w}\mathbf{x}), \quad (64)$$

where

$$w = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$$

is the 2×2 envelope matrix determined from the matrix envelope equation

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^\dagger w^{-1} \left(w^{-1}\right)^\dagger. \quad (65)$$

Because I_{CS} is an invariant of the particle dynamics, any function of I_{CS} is a solution of the Vlasov equation (1). However, in order to solve for the nonlinear Vlasov-Maxwell equations (1) and (2), the distribution function must generate the coupled linear space-charge force of the form in Eq. (63) as well. To achieve this goal, we select the distribution function to be the following generalized KV distribution:

$$f_{KV} = \frac{N_b |w|}{A\varepsilon\pi} \delta\left(\frac{I_{CS}}{\varepsilon} - 1\right). \quad (66)$$

Here, N_b and ε are constants, where N_b is the line density, and ε is the transverse emittance. Moreover, $|w|$ is the determinant of the envelope matrix w, and A is the area of the beam cross-section determined by $|w|$ and ε. Both $|w|$ and A are functions of $s = \beta_b ct$. The beam density profile in transverse configuration space is

$$n(x,y,s) = \int d\dot{x}d\dot{y}f_{KV} = \int d\left(\frac{r^2}{\varepsilon}\right)\frac{N_b}{A}\delta\left(\frac{I_{CS}}{\varepsilon}-1\right) = \begin{cases} N_b/A, & 0 \le \mathbf{x}^T w^{-1} w^{-1T}\mathbf{x} < \varepsilon\,, \\ 0, & \varepsilon < \mathbf{x}^T w^{-1} w^{-1T}\mathbf{x}\,. \end{cases} \tag{67}$$

In Eq. (67), the velocity integration with respect to $dv_x dv_y$ is calculated in the new coordinates (p,q) defined as

$$p \equiv w_1 v_x + w_2 v_y - \dot{w}_1 x - \dot{w}_2 y, \tag{68}$$

$$q \equiv w_3 v_x + w_4 v_y - \dot{w}_3 x - \dot{w}_4 y\,, \tag{69}$$

$$r^2 \equiv p^2 + q^2\,, \tag{70}$$

and the volume element transformation is

$$dv_x dv_y = \frac{1}{|w|}dpdq = \frac{2\pi}{|w|}rdr\,.$$

The density profile $n(x,y,s)$ obtained in Eq. (67) is indeed of the desired form. That is, $n(x,y,s)$ is constant inside the ellipse defined by

$$\mathbf{x}^\dagger \beta^* \mathbf{x} = \varepsilon\,,\ \ \beta^* \equiv w^{-1} w^{-1\dagger}, \tag{71}$$

and $n(x,y,s) = 0$ outside the ellipse. The ellipse defined by Eq. (71) is pulsating and rotating. Its transverse dimensions $a(s)$ and $b(s)$ and tilt angle $\theta(s)$ depend on $s = \beta_b ct$ and are determined from the matrix β^*. Because β^* is obviously real, symmetric, and positive definite, the two eigenvectors v_1 and v_2 of β^* are orthogonal with two positive eigenvalues λ_1 and λ_2. It is an elementary result[32] that the transverse dimensions of the ellipse are given by $a = \sqrt{\varepsilon/\lambda_1}$ and $b = \sqrt{\varepsilon/\lambda_2}$, and the tilt angle θ is that of v_1. The principal axis theorem[32] states that the diagonalizing matrix Q of β^* can be constructed as $Q = (v_1, v_2)$ with $Q^{-1} = Q^T$ and

$$Q^{-1}\beta^* Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{72}$$

We now introduce the rotating frame

$$\begin{pmatrix} X \\ Y \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{73}$$

The ellipse in (X,Y) coordinates is given

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1\,, \tag{74}$$

and the self-field force is

$$-\begin{pmatrix} \partial\psi/\partial X \\ \partial\psi/\partial Y \end{pmatrix} = \frac{2K_b}{a+b}\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{75}$$

Transforming back to the (x,y) coordinate, the self-field force can be expressed as

$$-\begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \end{pmatrix} = -\kappa_s\begin{pmatrix} x \\ y \end{pmatrix},\ \ \kappa_s = \frac{-2K_b}{a+b}Q\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}Q^{-1}. \tag{76}$$

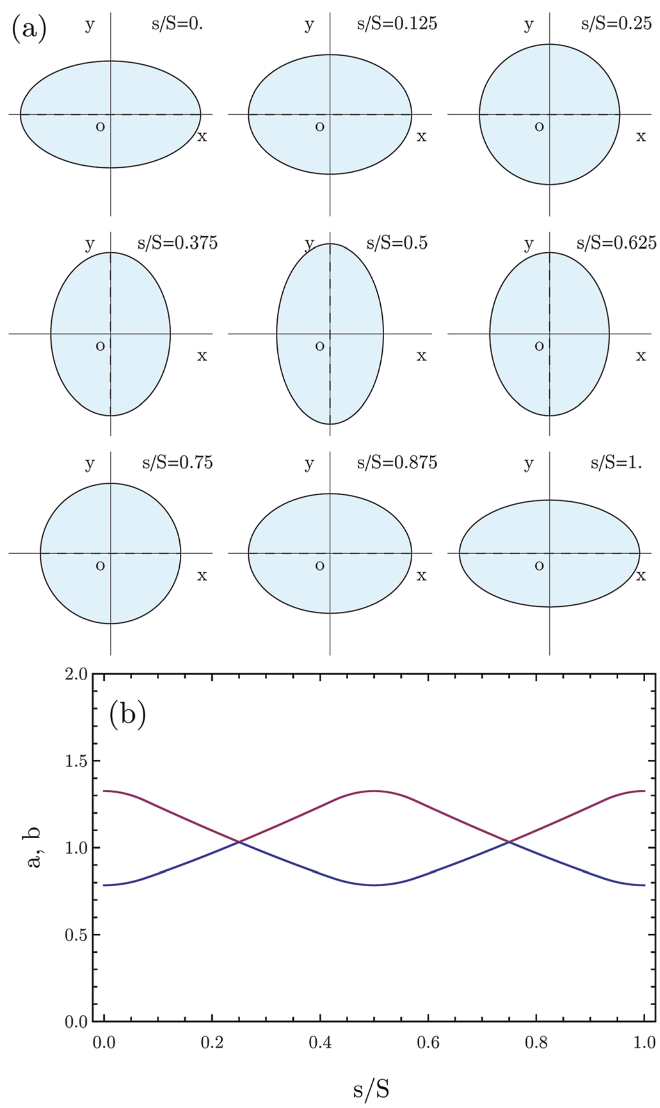

FIG. 4. (Color online) Matched solution in a standard FODO lattice, which is a two-rolling lattice. Beam cross-section (a) and major and minor dimensions (b) are plotted as functions of $s/S = \beta_b ct/S$ over the interval $0 \le s/S \le 1$. The beam cross-section pulsates without rotation. The transverse dimensions are normalized by $\sqrt{\varepsilon S}$.

The coupled linear space-charge coefficient κ_s is a function of the envelope matrix w and the constant emittance ε. When Eq. (76) is substituted back into Eq. (63), the envelope equation (65) becomes a closed nonlinear matrix equation for the envelope matrix w. Therefore, we have succeeded in finding a class of self-consistent solutions of the nonlinear Vlasov-Maxwell equations for high-intensity beams in a coupled transverse focusing lattice. The solution reduces to a nonlinear matrix ordinary differential equation for the envelope matrix w, which determines the geometry of the pulsating and rotating beam ellipse. The matrix envelope equation (65) can be numerically solved in a straightforward manner. Generalized KV solution can also be constructed using other generalized CS invariants of Eq. (59) with the help of the block Cholesky decomposition. This will be addressed in a future publication.

In 1979, Gluckstern[11] first derived a self-consistent KV distribution for the strongly coupled system with continuously rotated quadrupole field. However, as pointed out by

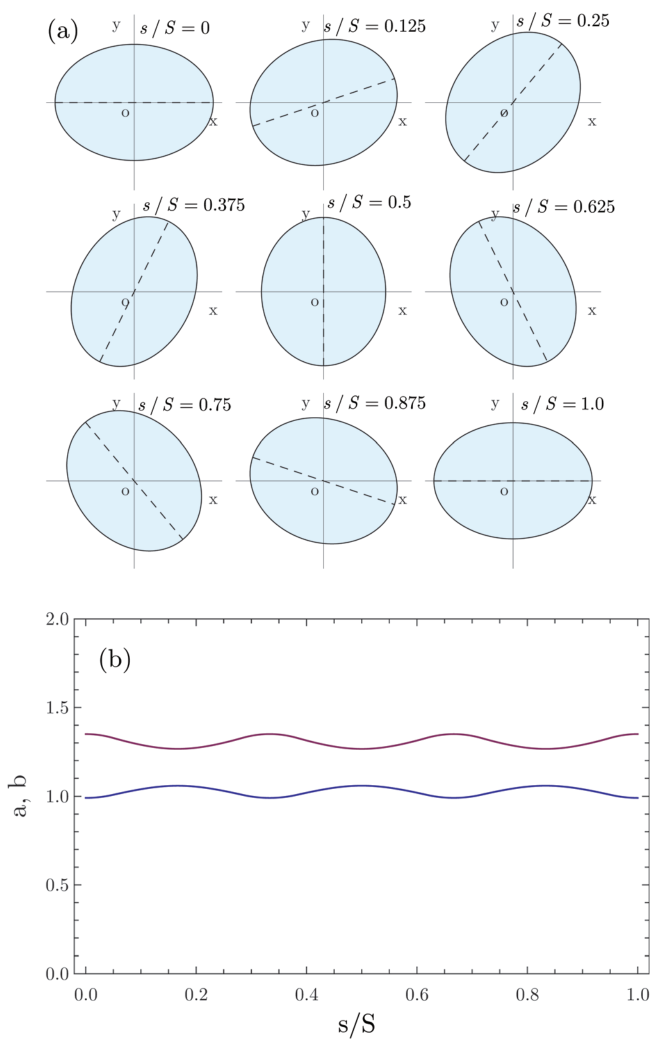

FIG. 5. (Color online) Matched solution in a three-rolling lattice. Beam cross-section (a) and major and minor dimensions (b) are plotted as functions of $s/S = \beta_b ct/S$ over the interval $0 \leq s/S \leq 1$. The beam cross-section pulsates and rotates. However the amplitude of the transverse pulsations reduces in comparison to the standard FODO lattice (two-rolling lattice). The transverse dimensions are normalized by $\sqrt{\varepsilon S}$.

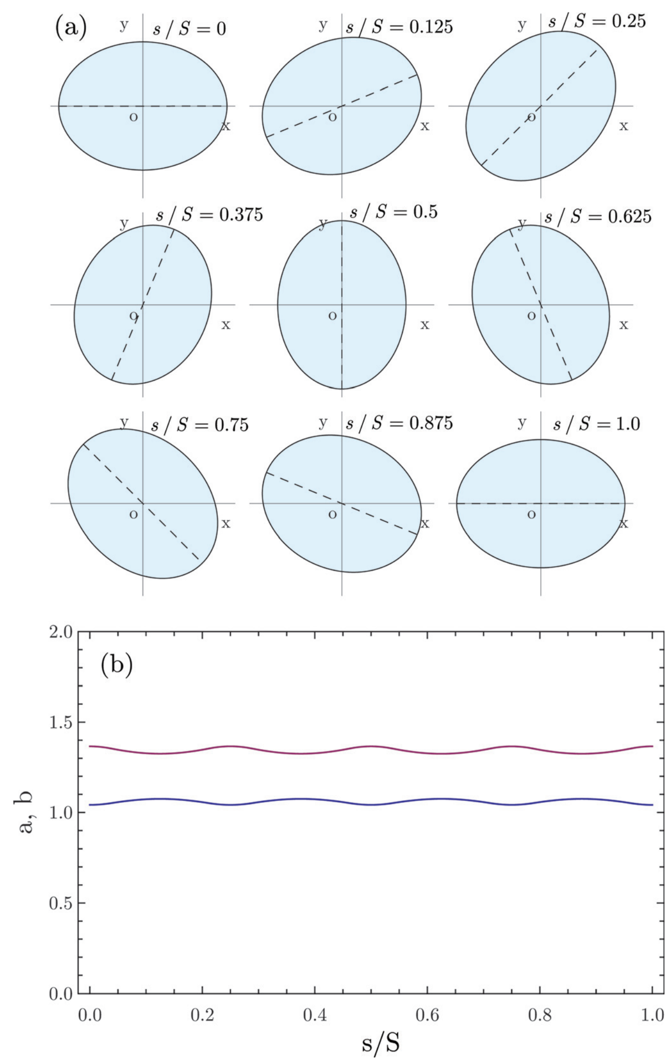

FIG. 6. (Color online) Matched solution in a four-rolling lattice. Beam cross-section (a) and major and minor dimensions (b) are plotted as functions of $s/S = \beta_b ct/S$ over the interval $0 \leq s/S \leq 1$. The beam cross-section pulsates and rotates. The amplitude of the transverse pulsations is reduced significantly in comparison to the standard FODO lattice (two-rolling lattice). The beam cross-section is asymptotic to a rigid rotor profile as the number of magnets increases. The transverse dimensions are normalized by $\sqrt{\varepsilon S}$.

Gluckstern, this method is only valid for this special focusing field. It does not apply to other coupled focusing field configurations such as the N-rolling lattice. On the other hand, the method developed in the present study is applicable to any coupled focusing field. Chernin[13] also found a KV distribution solution using a 4×4 transfer matrix [Eqs. (5)–(7) of Ref. 13] and found the equation for the evolution of the second order averages in closed form [Eqs. (3), (5), and (8) in Ref. 13]. To find a matched solution to these equations Chernin proposed an iterative algorithm to find the solution numerically. In Ref. 13, the case of continuously rotated quadrupoles was also studied, and the analytical matched solution was constructed using Gluckstern's method, instead of using the iterative numerical solution. In comparison, our envelope equation (65) is a closed equation for the envelope matrix which determines the generalized KV distribution.

The generalized CS theory and KV solution and the associated envelope equation can be used to study both weakly coupled system and strongly coupled system. An undesirable misalignment in a standard FODO lattice will result in a weakly coupled system, which has been studied in Ref. 18. In this paper, we will investigate the strongly coupled system with the N-rolling lattices described in Sec. II. We will compare the cases of strongly coupled three-rolling lattice and four-rolling lattice with the uncoupled FODO lattice (two-rolling lattice). In all three cases presented here, the normalized quadrupole focusing field is $\kappa_q \equiv q_b B'_q/\gamma_b m \beta_b c^2 = 15$. However, the filling factor for each magnet is reduced proportionally according to the number of magnets in the lattice, such that the total filling factor for the entire lattice is 30% for all three cases. The normalized self-field perveance is $K_b/\varepsilon = 0.1$. The matrix envelope equation (65) has been solved numerically to find a matched solution for each case. The numerical results plotted in Figs. 4–6 are the beam cross-sections and major and minor radii plotted a function of $s/S = \beta_b ct/S$ for the three cases, respectively. The transverse dimensions are normalized by $\sqrt{\varepsilon S}$. The dynamics of the beam pulsation and rotation are clearly demonstrated. By comparing the three cases, we see that the strong coupling in the three-rolling and four-rolling lattices does not deteriorate the beam quality. Instead, the coupling induces beam rotation and reduces beam pulsation. The oscillation amplitude of the beam dimensions decreases as the number of magnets increases. The reduction of oscillation amplitude for the four-rolling lattice, in comparison with the FODO (two-rolling) lattice, is remarkable. With a five-rolling lattice (not shown), we can achieve nearly constant beam radius, corresponding to a rigid rotor beam profile.

VI. CONCLUSIONS

In this paper, the CS theory and the KV distribution for high-intensity beams in an uncoupled focusing lattice have been generalized to the case of coupled transverse dynamics. The generalized CS theory has the same structure as the original CS theory for one degree of freedom. The four basic components of the original CS theory, i.e., the envelope equation, phase advance, transfer matrix, and the CS invariant, all have their counterparts in the generalized theory. The envelope function is generalized to an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are noncommutative. The generalized theory can provide a valuable framework for accelerator design and particle simulation studies for strongly and weakly coupled system. For example, it has been shown that the stability of coupled dynamics is completely determined by the generalized phase advance. Two stability criteria were given, which recover the known results about the sum and difference resonances in the weakly coupled limit.[19,20] In an uncoupled lattice, the KV distribution function, first analyzed in 1959, is the only known exact solution of the nonlinear Vlasov-Maxwell equations for high-intensity beams including self-fields in a self-consistent manner. The KV solution has been generalized to high-intensity beams in a coupled transverse lattice using the generalized CS invariant for coupled transverse dynamics. This solution projects to a rotating, pulsating elliptical beam in transverse configuration space, determined by the generalized matrix envelope equations. The fully self-consistent solution reduces the nonlinear Vlasov-Maxwell equations to a nonlinear matrix ordinary differential equation for the envelope matrix w, which determines the geometry of the pulsating and rotating beam ellipse. This result provides us with a new theoretical tool to investigate the dynamics of high-intensity beams in a coupled transverse lattice. In particular, we have designed and studied a type of strongly couple lattice, the so-called N-rollling lattice, which consists of N equally spaced quadrupole magnets. Each magnet rotates by an angle of π/N relative to its predecessor. It is found that strong coupling does not deteriorate the beam quality. Instead, the coupling induces beam rotation and reduces the beam pulsation. The oscillation amplitude of the transverse beam dimensions decreases for increasing number of magnets, and the beam cross-section is asymptotic to a rigid rotor profile as the number of magnets increases.

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