Kinetic Description of Nonlinear Wave and Soliton Excitations in Coasting Charged Particle Beams*

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Abstract

A one-dimensional kinetic model based on the Vlasov-Maxwell equations is used to describe nonlinear wave and soliton excitations in coasting charged particle beams. Analytical solutions are obtained for nonlinear traveling wave pulses, and particle-in-cell simulations are presented that describe stability properties and long-time evolution.

INTRODUCTION

High energy accelerators and transport systems [1] have a wide variety of applications ranging from basic research in high energy and nuclear physics, to ion-beam-driven high energy density physics and fusion. While progress has been made in 3D analytical and advanced numerical studies of intense charged particle beam propagation, it is important to develop an improved understanding of the collective processes and nonlinear dynamics. There is considerable interest in the development and application of simplified one-dimensional kinetic models to describe the nonlinear longitudinal dynamics of long coasting beams [2-6]. The present paper makes use of the one-dimensional kinetic model recently developed by Davidson and Startsev [5, 6] to describe nonlinear wave and soliton excitations in coasting charged particle beams. This kinetic description incorporates an improved g-factor model that includes the effects of transverse density profile shape at moderate beam intensities. In the present paper, the nonlinear evolution of wave and soliton excitations is examined for disturbances moving both faster and slower than the effective sound speed, incorporating the important effects of wave dispersion. Analytical solutions are obtained for nonlinear traveling wave pulses, and the results of particle-in-cell simulations are presented that describe the stability properties and long-time evolution.

THEORETICAL MODEL

We adopt the one-dimensinal kinetic model developed in [5, 6] that describes the self-consistent nonlinear evolution of the longitudinal distribution function $F_b(z, p_z, t)$, the average self-generated axial electric field $< E_z > (z, t)$, and the line density $\lambda_b(z, t) = \int dp_z F_b(z, p_z, t)$. Here, (z, p_z, t) denote variables in the beam frame, and it is assumed that the beam intensity is sufficiently low that the beam edge r_b and rms radius $R_b =< x^2 + y^2 >^{1/2}$ exhibit a negligible dependence on line density λ_b. Finally, the axial spatial variation in the number density $n_b(x, y, z, t)$ and the line density $\lambda_b(z, t)$ is assumed to be sufficiently slow that $k_z^2 r_w^2 \ll 1$, where $\partial/\partial z \sim k_z \sim L_z^{-1}$ is the inverse length scale of z-variations.

Within the context of this model we envision a nonlinear disturbance moving with characteristic pulse speed $v_p = const.$ Transforming from beam-frame variables (z, p_z, t) to wave-frame variables (z', p'_z, t'), where $z' = z - v_p t$, $p'_z = p_z - m_b v_p$ and $t' = t$, the kinetic equation for $F_b(z', p'_z, t')$ and $< E_z > (z', t') = -(\partial/\partial z')\Psi(z', t')$ can be expressed as

$$\frac{\partial}{\partial t'} F_b + v'_z \frac{\partial}{\partial z'} F_b - e_b \frac{\partial \Psi}{\partial z'} \frac{\partial}{\partial p'_z} F_b = 0, \tag{1}$$

$$e_b \Psi = m_b v_{b0}^2 \eta_b + m_b v_{b2}^2 r_w^2 \frac{\partial^2}{\partial z'^2} \eta_b, \tag{2}$$

where the constant velocities v_{b0} and v_{b2} are defined by

$$v_{b0}^2 = \frac{e_b^2 \lambda_{b0} g_0}{m_b}, \quad v_{b2}^2 = \frac{e_b^2 \lambda_{b0} g_2}{m_b}, \tag{3}$$

and η_b is the (dimensionless) line-density perturbation

$$\eta_b = \frac{\delta \lambda_b}{\lambda_{b0}} = \frac{\lambda_b - \lambda_{b0}}{\lambda_{b0}}. \tag{4}$$

The constant geometric factors g_0 and g_2 are defined in [5, 6] for a wide range of axi-symmetric density profiles n_b, and $\lambda_{b0} = \int dp'_z F_{b0} = const.$ is identified with the ambient line density in the absence of perturbation ($\Psi = 0 = \eta_b$ and $\partial/\partial z' = 0$). In Eq. (4), $\lambda_b(z', t') = \int dp'_z F_b(z', p'_z, t')$ is the line density, where the distribution function $F_b(z', p'_z, t')$ evolves according to Eq. (1).

NONLINEAR TRAVELING-WAVE AND SOLITON SOLUTIONS

Equations (1) and (2) can be used to investigate the detailed nonlinear evolution of the system in the wave frame for a wide variety of initial distributions $F_b(z', p'_z, 0)$, including the use of advanced numerical simulations. In this section, we illustrate that the Eqs. (1) and (2) support a broad range of nonlinear traveling-wave and soliton-like solutions analogous to Bernstein-Greene-Kruskal (BGK) solutions in electrically-neutral plasmas. Solutions are time-stationary in the wave frame ($\partial F_b/\partial t' = 0 = \partial \Psi/\partial t'$), and the corresponding distribution $F_b(z', p'_z)$ depends exclusively on the phase-space variables (z', p'_z) through the single-particle Hamiltonian H' defined by

$$H' = \frac{1}{2m_b} p_z'^2 + e_b \Psi(z'), \tag{5}$$

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where H' is a constant of the motion with $dH'/dt' = 0$. Here, $\Psi(z')$ is defined by Eq. (2), where $\eta_b(z') = [\lambda_b(z') - \lambda_{b0}]/\lambda_{b0}$ and $\lambda_b(z') = \int dp_z' F_b(H')$.

For $\partial/\partial t' = 0$, two classes of particles are distinguished. Theses are *untrapped* (or 'passing') particles that are not reflected by the potential $\Psi(z')$, and *trapped* (or 'reflected') particles that are reflected by $\Psi(z')$. The total line density is $\lambda_b(z') = \lambda_b^U(z') + \lambda_b^T(z')$, where

$$\lambda_b^U(z') = \left(\frac{m_b}{2}\right)^{1/2} \times \int_{[e_b\Psi]_{max}}^{\infty} dH' \frac{[F_b^{U<}(H') + F_b^{U>}(H')]}{(H' - e_b\Psi)^{1/2}}, \tag{6}$$

$$\lambda_b^T(z') = \left(\frac{m_b}{2}\right)^{1/2} \times \int_{e_b\Psi}^{[e_b\Psi]_{max}} dH' \frac{F_b^T(H')}{(H' - e_b\Psi)^{1/2}}. \tag{7}$$

In Eqs. (6) and (7), $[e_b\Psi]_{max}$ is the maximum value of $e_b\Psi(z')$, $F_b^{U<}(H')$ $[F_b^{U>}(H')]$ is the distribution of left traveling (right-traveling) untrapped particles with $v_z' = v_z - v_p < 0$ ($v_z' = v_z - v_p > 0$), and $F_b^T(H')$ is the distribution of trapped particles. It is clear that there is wide latitude in the specification of the untrapped-and trapped-particle distributions used for determination of the corresponding self-consistent potential $\Psi(z')$ from Eq. (2).

As an example, we consider the special case where there are no trapped particles ($F_b^T = 0$), and no right-moving untrapped particles ($F_b^{U>} = 0$), and take the distribution of the left-moving untrapped particles to be

$$F_b^{U<}(H') = \left(\frac{2H_0}{m_b}\right)^{1/2} \lambda_{b0}\delta(H' - H_0), \tag{8}$$

where $H_0 \equiv m_b v_p^2/2$ and v_p is the (positive) pulse speed relative to the beam frame. From Eqs. (6)–(8), it readily follows that $\lambda_b^T(z') = 0$ and

$$\lambda_b^U(z') = \frac{\lambda_{b0}}{(1 - e_b\Psi/H_0)^{1/2}}. \tag{9}$$

Equation (9) gives

$$\frac{e_b\Psi}{H_0} = 1 - \frac{1}{(1+\eta_b)^2}, \tag{10}$$

which expresses $e_b\Psi/H_0$ in terms of $\eta_b = (\lambda_b^U - \lambda_{b0})/\lambda_{b0}$.

Equation (10) can be substituted into Eq. (2) to give a closed nonlinear equation for $\eta_b(z')$. We introduce the Mach number *M* defined by

$$M = v_p/v_{b0}, \tag{11}$$

where v_p is the pulse speed, and $v_{b0} = (e_b^2\lambda_{b0}g_0/m_b)^{1/2}$ is the effective sound speed. Substituting Eq. (10) with $H_0 = m_b v_p^2/2$ into Eq. (2) and rearranging terms gives

$$\frac{v_{b2}^2}{v_{b0}^2} r_w^2 \frac{\partial^2}{\partial z'^2}\eta_b = -\frac{\partial}{\partial \eta_b} V(\eta_b), \tag{12}$$

where $V(\eta_b)$ is the effective potential defined by

$$V(\eta_b) = \frac{1}{2}\eta_b^2 \left(\frac{1 - M^2 + \eta_b}{1 + \eta_b}\right), \tag{13}$$

and $\eta_b = (\lambda_b - \lambda_{b0})/\lambda_{b0} > -1$ is assumed.

For purposes of illustration, we consider values of M^2 close to unity, in which case Eq. (12) supports weakly nonlinear solutions consistent with $|\eta_b| < 1$ and $r_w^2\partial^2/\partial z'^2 \sim k_z^2 r_w^2 < 1$. Expanding for small η_b, Eq. (12) gives

$$\frac{v_{b2}^2}{v_{b0}^2} r_w^2 \frac{\partial^2}{\partial z'^2}\eta_b = -(1 - M^2)\eta_b - \frac{3}{2}M^2\eta_b^2 \tag{14}$$

correct to quadratic order. Two cases can be distinguished in analyzing Eq. (14). For $M^2 = 1 + \epsilon$ for small $\epsilon > 0$, the solution to Eq. (14) can be expressed as

$$\eta_b(z') = \left(\frac{M^2 - 1}{M^2}\right) \times sech^2\left[\frac{1}{2}(M^2 - 1)^{1/2}\frac{v_{b0}}{v_{b2} r_w}(z - v_p t)\right], \tag{15}$$

where $z' = z - v_p t$. Equation (15) is the familiar soliton solution encountered in the analysis of the Korteweg-deVries equation, which corresponds to an isolated pulse traveling with velocity v_p in the beam frame.

For $M^2 = 1 - \epsilon$ for small $\epsilon > 0$, Eq. (14) supports nonlinear periodic traveling wave solutions for $\eta_b(z - v_p t)$. For sufficiently small amplitude, we neglect the quadratic term in Eq. (14), which gives

$$\eta_b(z') = \hat{\eta}_b \cos[k_0(z - v_p t) + \phi_0], \tag{16}$$

where $\hat{\eta}_b$ and ϕ_0 are constant, and

$$k_0^2 r_w^2 \frac{v_{b2}^2}{v_{b0}^2} = (1 - M^2) = 1 - \frac{v_p^2}{v_{b0}^2}. \tag{17}$$

Making the identification $v_p = \omega_0/k_0$, where ω_0 and k_0 are the frequency and wavenumber of the traveling wave in the beam frame, Eq. (17) gives

$$\omega_0^2 = k_0^2 v_{b0}^2 - k_0^4 r_w^2 v_{b2}^2, \tag{18}$$

where $k_0^2 r_w^2 \ll 1$ is assumed. Note that Eqs. (16) and (17) correspond to a wave disturbance moving with phase velocity ω_0/k_0 slightly slower than the sound speed v_{b0}.

SIMULATION RESULTS

The one-dimensional kinetic g-factor model in Eqs. (1)–(4) has been implemented in the nonlinear code BEST [1]. We have carried out initial numerical studies of nonlinear wave propagation in such a system, confirming the existence of solitons when $M > 1$ [Figs. 1 and 2]. In the simulations presented here, the initial distribution function is taken to be a shifted Maxwellian in the beam frame with

$$F_{b0}(z, p_z) = \frac{1}{(2\pi m_b k_b T)^{1/2}}\left[\exp\left\{-\frac{p_z^2}{2m_b k_b T}\right\} + \eta_{b0}(z)\exp\left\{-\frac{[p_z - m_b v_{b0}\eta_{b0}(z)]^2}{2m_b k_b T}\right\}\right], \tag{19}$$

where the thermal spread is $v_{th}/v_{b0} = 0.1$, and $\eta_{b0} \ll 1$ is assumed. For such a small thermal spread, the beam could be considered cold. Therefore, the evolution of a small-amplitude density perturbation $|\eta_b(z,t)| \ll 1$ could be well approximated by a Korteweg-deVris equation [6]. Figure 1 shows the time evolution of an initial sinusoidal perturbation with nonlinearity parameter $\eta_{b0}^{max} = \delta\lambda_b^{max}/\lambda_{b0} = 0.05$ at $t = 0$ [curve (a)] and dispersion parameter $v_{b2}r_w/v_{b0}L = 10^{-2}$. As time progresses, the perturbation profile steepens due to the nonlinearity [curve (b)]. When the front steepening is stabilized by dispersion, a modulational instability at the back of the wavefront develops [curve (c)], and results in the formation of a train of solitons with different amplitudes [curve (d)]. This type of behavior was also observed in the original numerical simulations of the Korteweg-deVries equation by Zabusky and Kruskal [7].

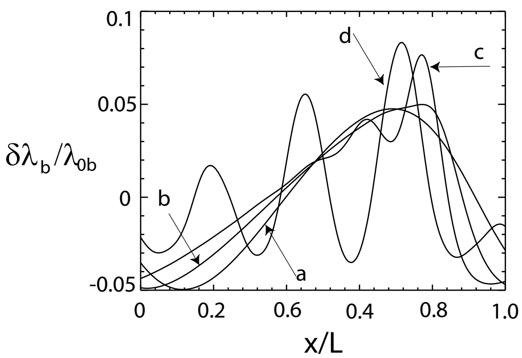

Figure 1: An initial sinusoidal perturbation with $\eta_{b0}^{max} = \delta\lambda_b^{max}/\lambda_{b0} = 0.05$ and $v_{b2}r_w/v_{b0}L = 10^{-2}$ (a) steepens due to the nonlinearity (b). A modulational instability at the back of the wavefront develops (c), and results in the formation of a train of solitons with different amplitudes (d).

Figure 2 shows the time evolution of a perturbation initially in the form given by Eq. (15) but with an amplitude twice too large, i.e., $\eta_{b0}^{max} = \delta\lambda_b^{max}/\lambda_{b0} = 0.08$ and $v_{b2}r_w/v_{b0}L = 5 \times 10^{-3}$. As time progresses, the initial perturbation splits into two solitons and a small-amplitude oscillatory perturbation [curves (b) and (c)]. This behavior can be expected from the general solution of Korteweg-deVries equation, which identifies the number of solitons as $t \to \infty$ as the number of bound states in the quantum-mechanical potential $V(z) = -\eta_{b0}(z)/6$, where $\eta_{b0}(z) \equiv \eta_b(z, t = 0)$ is the initial perturbation. The oscillatory remainder in Fig. 2 would be exactly zero if the quantum-mechanical potential $V(z)$ was reflectionless.

CONCLUSIONS

A one-dimensional kinetic model based on the Vlasov-Maxwell equations has been used to describe nonlinear wave and soliton excitations in coasting charged particle

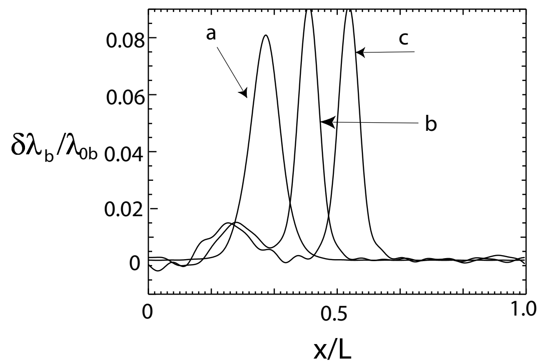

Figure 2: An initial perturbation in the form given by Eq. (15) but with twice larger amplitude $\eta_{b0}^{max} = \delta\lambda_b^{max}/\lambda_{b0} = 0.08$ and $v_{b2}r_w/v_{b0}L = 5 \times 10^{-3}$ (a) splits into two solitons, (b) and (c), and a small-amplitude oscillatory perturbation.

beams. Analytical solutions were obtained for nonlinear traveling wave pulses, and initial particle-in-cell simulations were presented that describe stability properties and long-time evolution.

REFERENCES

[1] R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, 2001), and references therein.

[2] O. Boine-Frankenheim, I. Hofmann and G. Rumulo, Phys. Rev. Lett. **82**, 3256 (1999).

[3] L. K. Spentzouris, J. -F. Ostiguy and P. L. Colestock, Phys. Rev. Lett. **76**, 620 (1996)

[4] H. Schamel and R. Fedele, Phys. Plasmas **7**, 3421 (2000).

[5] R. C. Davidson and E. A. Startsev, Phys. Rev. ST Accel. Beams. **7**, 024401 (2004).

[6] R. C. Davidson, Phys. Rev. ST Accel. Beams. **7**, 054402 (2004).

[7] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).