Three-dimensional kinetic stability theorem for high-intensity charged particle beams

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Global conservation constraints obtained from the nonlinear Vlasov–Maxwell equations are used to derive a three-dimensional kinetic stability theorem for an intense non-neutral ion beam (or charge bunch) propagating in the z direction with average axial velocity $v_b=$const and characteristic kinetic energy $(\gamma_b-1)mc^2$ in the laboratory frame. Here, $\gamma_b=(1-v_b^2/c^2)^{-1/2}$ is the relativistic mass factor, and a perfectly conducting cylindrical wall is located at radius $r=r_w$, where $r=(x^2+y^2)^{1/2}$ is the radial distance from the beam axis. The particle motion in the beam frame ('primed' coordinates) is assumed to be nonrelativistic, and the beam is assumed to have sufficiently high directed axial velocity that $v_b \gg |\mathbf{v}'|$. Space-charge effects and transverse electromagnetic effects are incorporated into the analysis in a fully self-consistent manner. The nonlinear Vlasov–Maxwell equations are Lorentz-transformed to the beam frame, and the applied focusing potential is assumed to have the (time-stationary) form $\psi'_{\rm sf}(\mathbf{x}')=(\gamma_b m/2)[\omega_{\beta\perp}^2(x'^2+y'^2)+\omega_{\beta z}^2 z'^2]$, where $\omega_{\beta\perp}$ and $\omega_{\beta z}$ are constant focusing frequencies. It is shown that a sufficient condition for linear and nonlinear stability for perturbations with arbitrary polarization about a beam equilibrium distribution $f_{\rm eq}(\mathbf{x}',\mathbf{p}')$ is that $f_{\rm eq}$ be a monotonically decreasing function of the single-particle energy, i.e., $\partial f_{\rm eq}(H')/\partial H' \leq 0$. Here, $H'=\mathbf{p}'^2/2m+\psi'_{\rm sf}(\mathbf{x}')+\phi_{\rm eq}(\mathbf{x}')$, where $\phi_{\rm eq}(\mathbf{x}')$ is the space-charge potential. © *1998 American Institute of Physics.* [S1070-664X(98)01909-0]

I. INTRODUCTION

Periodic focusing accelerators[1–3] and transport systems have a wide range of applications ranging from basic scientific research, to applications[4–6] such as heavy ion fusion, tritium production, spallation neutron sources, and nuclear waste treatment. Of particular importance, at the high beam currents and charge densities of practical interest, are the effects of the intense self fields produced by the beam space charge and current on determining detailed equilibrium, stability, and transport properties. While considerable progress can be made in understanding the self-consistent evolution of the beam distribution function $f_b(\mathbf{x},\mathbf{p},t)$ and the electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ in kinetic analyses[1,7–9] based on the nonlinear Vlasov–Maxwell equations, the effects of finite geometry, space-charge effects, and beam emittance generally make predictions of detailed stability behavior difficult. It is therefore important to develop a basic understanding of the class of distribution functions f_b that are *stable* and can propagate quiescently over large distances, even in parameter regimes where space-charge effects are intense and play a controlling role in the nonlinear beam dynamics and transport properties.

With this in mind, the present analysis makes use of global (spatially averaged) conservation constraints[10] satisfied by the nonlinear Vlasov–Maxwell equations to determine a sufficient condition for stability of an intense non-neutral ion beam (or charge bunch) propagating in the z direction with average axial velocity $v_b=$const along the axis of a perfectly conducting cylindrical pipe with wall radius $r=(x^2+y^2)^{1/2}=r_w$. The theoretical approach used in the present analysis is motivated by the early work of Newcomb,[11] Gardner,[12] and Fowler,[13] carried out for perturbations about a spatially uniform, electrically neutral, nonrelativistic plasma, and by the stability theorem developed by Davidson and Krall[10,14] for a one-component non-neutral plasma column confined radially by a uniform axial magnetic field. The present analysis represents a major generalization of the stability theorem developed in Ref. 10, which was carried out for small-amplitude electrostatic perturbations about a nonrelativistic non-neutral plasma column which is infinite in axial extent. In particular, using global conservation constraints, the instability theorem developed here is fully nonlinear and electromagnetic, and applies to three-dimensional perturbations with arbitrary amplitude and polarization about a finite-length charge bunch propagating with relativistic axial velocity $v_b\hat{\mathbf{e}}_z$.

To briefly summarize, the present analysis considers an intense charged-particle beam consisting of positively charged ions with charge $+Z_ie$ and rest mass m propagating in the positive z direction with average axial v_b and characteristic kinetic energy $(\gamma_b-1)mc^2$ in the laboratory frame, where $\gamma_b=(1-v_b^2/c^2)^{-1/2}$ is the relativistic mass factor. A perfectly conducting cylindrical wall is located at radius $r=r_w$, where $r=(x^2+y^2)^{1/2}$ is the radial distance from the beam axis. The particle motion in the beam frame (the 'primed' frame) is assumed to be nonrelativistic with $|\mathbf{v}'|\ll c$, and the beam is assumed to have sufficiently high directed axial velocity that $|\mathbf{v}'|\ll v_b$. The beam current and charge density are allowed to be sufficiently intense that the characteristic electrostatic energy $Z_ie\phi$ of a beam particle is comparable in magnitude with the characteristic kinetic energy of a particle in the beam frame. Therefore, collective processes associated with space-charge effects and self-consistent changes in the beam current can play a controlling role in the nonlinear evolution of the distribution of beam particles $f_b(\mathbf{x},\mathbf{p},t)$ in the six-dimensional phase space $(\mathbf{x},\mathbf{p})$. Finally, it is assumed that transverse focusing of the beam particles is provided by the average effects of applied magnetic or electric focusing fields.

In more detail, the organization of this article is as follows. In Sec. II, the theoretical model and assumptions are discussed, and the nonlinear Vlasov–Maxwell equations and the applied focusing force are Lorentz-transformed to the beam frame (the primed frame) where the particle motion is assumed to be nonrelativistic. We then specialize in Sec. III to the case where the applied focusing potential in the beam frame (assumed stationary) is of the form $\psi'_{\rm sf}(\mathbf{x}')=(\gamma_b m/2)[\omega^2_{\beta\perp}(x'^2+y'^2)+\omega^2_{\beta z}z'^2]$, where $\omega_{\beta\perp}$ and $\omega_{\beta z}$ are constant focusing frequencies, and derive global (spatially averaged) conservation constraints corresponding to conservation of particle plus field energy, and generalized entropy. Introducing a generalized Helmholtz free energy $F'(t')=$const, the global conservation constraints are used in Sec. IV to derive a three-dimensional kinetic stability theorem for perturbations with arbitrary polarization. It is shown that a *sufficient condition for linear and nonlinear stability* for perturbations about a beam equilibrium distribution function $f_{\rm eq}(\mathbf{x}',\mathbf{p}')$ is that $f_{\rm eq}$ be a monotonically decreasing function of the single-particle energy H' [Eq. (33)],

$$\frac{\partial}{\partial H'}\, f_{\rm eq}(H')\leqslant 0.$$

Here, $H'=\mathbf{p}'^2/2m+\psi'_{\rm sf}(\mathbf{x}')+\phi_{\rm eq}(\mathbf{x}')$ is defined in Eq. (30), where $\phi_{\rm eq}(\mathbf{x}')$ is the equilibrium space-charge potential. For completeness, in Sec. V we show that the thermal equilibrium distribution $f_{\rm eq}=g(H')\equiv\beta'\exp(-H'/T'_b)$, where β' and T'_b are positive constants, is the unique choice of distribution function that *minimizes* the classical Helmholtz free energy, $F'_H(t')=$const. The thermal equilibrium properties of a charge bunch are summarized for the two limiting regimes corresponding to a low-emittance, high-intensity beam with strong space-charge effects, and a low-intensity, high-emittance beam with weak space-charge fields.

Finally, it is very important to recognize the wide range of applicability of the three-dimensional stability theorem developed in the present analysis. Most notably, it applies to perturbations about beam equilibria $f_{\rm eq}(H')$ with arbitrary polarization and initial amplitude; to continuous beams that are radially confined and infinite in axial extent ($\omega_{\beta\perp}\neq 0$, $\omega_{\beta z}=0$); to charge bunches that are radially and axially confined ($\omega_{\beta\perp}\neq 0$ and $\omega_{\beta z}\neq 0$); and to beams with arbitrary space-charge intensity consistent with the applied focusing potential $\psi'_{\rm sf}(\mathbf{x}')$ providing confinement of the beam particles.

II. THEORETICAL MODEL AND ASSUMPTIONS

The present analysis considers an intense charged-particle beam consisting of positively charged ions with charge $+Z_ie$ and rest mass m propagating in the positive z direction with average axial v_b and characteristic kinetic energy $(\gamma_b-1)mc^2$ in the laboratory frame, where $\gamma_b=(1-v_b^2/c^2)^{-1/2}$ is the relativistic mass factor. A perfectly conducting cylindrical wall is located at radius $r=r_w$, where $r=(x^2+y^2)^{1/2}$ is the radial distance from the beam axis. It is assumed that $\nu/\gamma_b\ll 1$, where $\nu=N_bZ_i^2e^2/mc^2$ is Budker's parameter,[1] c is the speed of light *in vacuo*, $N_b=\int dx\,dy\,n_b$ is the number of beam particles per unit axial length, and $n_b(x,y,z,t)=\int d^3p f_b(\mathbf{x},\mathbf{p},t)$ is the particle density. The particle motion in the beam frame (the primed frame) is assumed to be nonrelativistic with $|\mathbf{v}'|\ll c$, and the beam is assumed to have sufficiently high directed axial velocity that

$$|\mathbf{v}'|\ll v_b. \tag{1}$$

Although $\nu/\gamma_b\ll 1$ is satisfied in the regimes of practical interest, the beam current and charge density are allowed to be sufficiently intense that the characteristic electrostatic energy $Z_ie\phi$ of a beam particle is comparable in magnitude with the characteristic kinetic energy of a particle in the beam frame. Therefore, collective processes associated with space-charge effects[1,7–9] and self-consistent changes in the beam current can play a controlling role in the nonlinear evolution of the distribution of beam particles $f_b(\mathbf{x},\mathbf{p},t)$ in the six-dimensional phase space $(\mathbf{x},\mathbf{p})$. Finally, it is assumed that transverse focusing of the beam particles is provided by applied magnetic or electric focusing fields. For example, for a periodic-focusing quadrupole magnetic field, assuming a thin beam with characteristic beam radius r_b small in comparison with the axial period S of the lattice, the applied *quadrupole field* $\mathbf{B}^0_{\rm qf}(\mathbf{x})$ in the laboratory frame can be approximated by[1]

$$\mathbf{B}^0_{\rm qf}(\mathbf{x})=B'_q(z)(y\hat{\mathbf{e}}_x+x\hat{\mathbf{e}}_y). \tag{2}$$

Here, (x,y) is the transverse displacement from the beam axis, and $B'_q(z)\equiv[\partial B_x/\partial y]_{(x,y)=(0,0)}=[\partial B_y/\partial x]_{(x,y)=(0,0)}$, where $B'_q(z+S)=B'_q(z)$. On the other hand, a model widely used in the *smooth-focusing* approximation corresponds to a transverse focusing electric field $\mathbf{E}^0_{\rm sf}(\mathbf{x})$ of the form[15]

$$\mathbf{E}^0_{\rm sf}(\mathbf{x})=-\frac{1}{Z_ie}\,m\omega^2_{\beta\perp}(x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y), \tag{3}$$

where $\omega_{\beta\perp}=$const is an effective betatron frequency for transverse oscillations. The focusing electric field in Eq. (3) would be produced by a (hypothetical) uniformly distributed, fixed charge background with charge density $\rho=-m\omega^2_{\beta\perp}/2\pi Z_ie=$const. Equation (3) is often used to model the *average* focusing properties of an alternating-gradient lattice of magnetic or electric quadrupoles.

The Vlasov–Maxwell equations[1] for the evolution of the distribution function $f_b(\mathbf{x},\mathbf{p},t)$ and self-consistent electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ provide a complete nonlinear description of the collective interaction of the beam particles with the applied and self-generated electric and magnetic fields. In this regard, analysis of the Vlasov–Maxwell equations is greatly simplified by transforming to a frame of reference moving at the average axial velocity v_b

=const of the beam particles, particularly because of the assumptions that the particle motion is nonrelativistic in the beam frame and that $|\mathbf{v}'| \ll v_b$. The Lorentz transformation relating the primed variables $(\mathbf{x}',\mathbf{p}',t')$ in the beam frame to the unprimed variables $(\mathbf{x},\mathbf{p},t)$ in the laboratory frame is given by[16]

$$\begin{aligned} &x'=x, \quad y'=y, \quad z'=\gamma_b(z-v_b t), \\ &p'_x=p_x, \quad p'_y=p_y, \quad p'_z=\gamma_b(p_z-\gamma m v_b), \\ &t'=\gamma_b(t-v_b z/c^2). \end{aligned} \tag{4}$$

Here, the particle momentum and velocity are related by $\mathbf{p}=\gamma m\mathbf{v}$ and $\mathbf{p}'=\gamma' m\mathbf{v}'$, where the kinematic mass factors $\gamma=(1+\mathbf{p}^2/m^2c^2)^{1/2}$ and $\gamma'=(1+\mathbf{p}'^2/m^2c^2)^{1/2}$ transform according to $\gamma'=\gamma_b(\gamma-v_b p_z/mc^2)$, where $\gamma_b=(1-v_b^2/c^2)^{-1/2}$. In the beam frame, the nonlinear Vlasov equation for the distribution function $f_b(x',\mathbf{p}',t')$ can be expressed as

$$\frac{\partial f_b}{\partial t'}+\mathbf{v}'\cdot\frac{\partial f_b}{\partial \mathbf{x}'}+\left[Z_i e\left(\mathbf{E}'+\frac{1}{c}\,\mathbf{v}'\times\mathbf{B}'\right)+\mathbf{F}'_{\rm foc}\right]\cdot\frac{\partial f_b}{\partial \mathbf{p}'}=0. \tag{5}$$

Here, $\mathbf{E}'(\mathbf{x}',t')$ and $\mathbf{B}'(\mathbf{x}',t')$ are the self-generated electric and magnetic fields in the beam frame, and $\mathbf{F}'_{\rm foc}=Z_i e(\mathbf{E}'_{\rm foc}+c^{-1}\mathbf{v}'\times\mathbf{B}'_{\rm foc})$ is the applied focusing force on a particle in the beam frame. Moreover, because the particle motion is assumed to be nonrelativistic in the beam frame, we approximate $\gamma'=1+\mathbf{p}'^2/2mc^2$ and $\mathbf{p}'=m\mathbf{v}'$ in the subsequent analysis of Eq. (5).

Maxwell's equations in the beam frame relate $\mathbf{E}'(\mathbf{x}',t')$ and $\mathbf{B}'(\mathbf{x}',t')$ self-consistently to the distribution function $f_b(\mathbf{x}',\mathbf{p}',t')$. In this regard, it is convenient to introduce the scalar and vector potentials, $\phi'(\mathbf{x}',t')$ and $\mathbf{A}'(\mathbf{x}',t')$, and express

$$\begin{aligned} &\mathbf{E}'=\mathbf{E}'_L+\mathbf{E}'_T=-\frac{\partial}{\partial x'}\,\phi'-\frac{1}{c}\,\frac{\partial}{\partial t'}\,\mathbf{A}', \\ &\mathbf{B}'=\frac{\partial}{\partial \mathbf{x}'}\times\mathbf{A}', \end{aligned} \tag{6}$$

where $\mathbf{E}'_L=-\nabla'\phi'$ is the longitudinal electric field, $\mathbf{E}'_T=-c^{-1}\partial\mathbf{A}'/\partial t'$ is the transverse electric field, and the Coulomb gauge condition with $\nabla'\cdot\mathbf{A}'=0$ is assumed. The Maxwell equations $\nabla'\cdot\mathbf{B}'=0$ and $\nabla'\times\mathbf{E}'=-c^{-1}\partial\mathbf{B}'/\partial t'$ are automatically satisfied by Eq. (6), and Poisson's equation and the $\nabla'\times\mathbf{B}'$ Maxwell equation are readily expressed in the beam frame as

$$\nabla'^2\phi'=-4\pi Z_i e\int d^3p' f_b, \tag{7}$$

$$\nabla'^2\mathbf{A}'=-\frac{4\pi}{c}\,Z_i e\int d^3p'\mathbf{v}' f_b+\frac{1}{c^2}\,\frac{\partial^2\mathbf{A}'}{\partial t'^2}+\frac{1}{c}\,\nabla'\,\frac{\partial\phi'}{\partial t'}, \tag{8}$$

where $\nabla'^2\equiv\partial^2/\partial x'^2+\partial^2/\partial y'^2+\partial^2/\partial z'^2$. Here, the electrostatic potential $\phi'(\mathbf{x}',t')$ is determined self-consistently in terms of the beam charge density $\rho'_b(\mathbf{x}',t')=Z_i e\int d^3p' f_b(\mathbf{x}',\mathbf{p}',t')$ by means of Eq. (7), and $\mathbf{A}'(\mathbf{x}',t')$ is determined in terms of the beam current density $\mathbf{J}'_b(\mathbf{x}',t')=Z_i e\int d^3p'\mathbf{v}' f_b(\mathbf{x}',\mathbf{p}',t')$ by means of Eq. (8). With regard to the boundary conditions at the perfectly conducting cylindrical wall, we impose the requirement that the tangential electric field and the normal magnetic field vanish at radius $r=r_w$. That is, $[E_z]_{r=r_w}=[E_\theta]_{r=r_w}=[B_r]_{r=r_w}=0$, where B_r, E_θ and E_z denote field components in cylindrical polar coordinates in the laboratory frame. In the beam frame, the corresponding field components are $E'_z=E_z$, $B'_r=\gamma_b(B_r+v_b E_\theta/c)$, and $E'_\theta=\gamma_b(E_\theta+v_b B_r/c)$, so that the corresponding boundary conditions at the conducting wall in the beam frame are also given by $[E'_z]_{r'=r_w}=[E'_\theta]_{r'=r_w}=[B'_r]_{r'=r_w}=0$. In terms of the scalar and vector potentials, $\phi'(\mathbf{x}',t')$ and $\mathbf{A}'(\mathbf{x}',t')$, these boundary conditions can be expressed in the equivalent form

$$\begin{aligned} \phi'(r'=r_w,\theta',z',t')&=A'_z(r'=r_w,\theta',z',t') \\ &=A'_\theta(r'=r_w,\theta',z',t')=0. \end{aligned} \tag{9}$$

In Eq. (9), without loss of generality, the constant values of ϕ', A'_z and A'_θ at $r'=r_w$ have been set equal to zero.

As a general remark, the components of potential $(\phi',\mathbf{A}')$ in the beam frame are related to the components of potential $(\phi,\mathbf{A})$ in the laboratory frame by[16]

$$\begin{aligned} &\phi'=\gamma_b(\phi-v_b A_z/c), \\ &A'_x=A_x, \quad A'_y=A_y, \\ &A'_z=\gamma_b(A_z-v_b\phi/c). \end{aligned} \tag{10}$$

Of course the inverse transformations of Eqs. (4) and (10) are obtained by interchanging primed and unprimed variables, and making the replacement $v_b\to -v_b$. For example, in the special case where $\phi'\neq 0$ and $\mathbf{A}'\simeq 0$ in the beam frame, it follows that $A_z\simeq(v_b/c)\phi$, $\mathbf{A}_\perp\simeq 0$, and $\phi\simeq\gamma_b\phi'$ in the laboratory frame.

We now turn to an evaluation of the applied focusing force $\mathbf{F}'_{\rm foc}=Z_i e(\mathbf{E}'_{\rm foc}+c^{-1}\mathbf{v}'\times\mathbf{B}'_{\rm foc})$ on a particle in the beam frame. Here, the applied electric and magnetic fields transform according to[16] $[\mathbf{E}']_{\rm foc}=[E_z\hat{\mathbf{e}}_z+\gamma_b(E_x\hat{\mathbf{e}}_x+E_y\hat{\mathbf{e}}_y)+\gamma_b c^{-1}v_b\hat{\mathbf{e}}_z\times\mathbf{B}]_{\rm foc}$ and $[\mathbf{B}']_{\rm foc}=[B_z\hat{\mathbf{e}}_z+\gamma_b(B_x\hat{\mathbf{e}}_x+B_y\hat{\mathbf{e}}_y)-\gamma_b c^{-1}v_b\hat{\mathbf{e}}_z\times\mathbf{E}]_{\rm foc}$. For the *smooth-focusing* electric field $\mathbf{E}^0_{\rm sf}(\mathbf{x})$ defined in Eq. (3), some straightforward algebra shows that the corresponding applied focusing force in the beam frame is

$$\begin{aligned} [\mathbf{F}'_{\rm foc}]_{\rm sf}=-\gamma_b m\omega^2_{\beta\perp}\Bigg[&\left(1+\frac{v'_z v_b}{c^2}\right)(x'\hat{\mathbf{e}}'_x+y'\hat{\mathbf{e}}'_y) \\ &-\frac{v_b}{c^2}\,(x'v'_x+y'v'_y)\hat{\mathbf{e}}'_z\Bigg]. \end{aligned} \tag{11}$$

To the level of accuracy of Eq. (1), because $|\mathbf{v}'|\ll c$ is assumed, we approximate Eq. (11) by

$$[\mathbf{F}'_{\rm foc}]_{\rm sf} = -\gamma_b m\omega^2_{\beta\perp}(x'\hat{\mathbf{e}}'_x + y'\hat{\mathbf{e}}'_y). \tag{12}$$

In Eqs. (11) and (12), $(\hat{\mathbf{e}}'_x, \hat{\mathbf{e}}'_y, \hat{\mathbf{e}}'_z)$ are unit Cartesian vectors in the beam frame. Similarly, some straightforward algebra carried out for the periodic *quadrupole focusing* magnetic field $\mathbf{B}^0_{\rm qf}(\mathbf{x})$ defined in Eq. (2) gives for the applied focusing force in the beam frame

$$\begin{aligned}[\mathbf{F}'_{\rm foc}]_{\rm qf} = {} & Z_i e\gamma_b B'_q[\gamma_b(z'+v_b t')] \\ & \times\left(\frac{1}{c}(v_b+v'_z)(-x'\hat{\mathbf{e}}'_x + y'\hat{\mathbf{e}}'_y) \right. \\ & \left. + \frac{1}{c}(x'v'_x - y'v'_y)\hat{\mathbf{e}}'_z\right).\end{aligned} \tag{13}$$

Here, $B'_z(z)$ has been evaluated at $z=\gamma_b(z'+v_b t')$. Similarly, for $|\mathbf{v}'| \ll v_b$, the quadrupole focusing force in Eq. (13) can be approximated by

$$[\mathbf{F}'_{\rm foc}]_{\rm qf} = -\frac{1}{c} Z_i e\gamma_b v_b B'_q[\gamma_b(z'+v_b t')](x'\hat{\mathbf{e}}'_x - y'\hat{\mathbf{e}}'_y). \tag{14}$$

Because $B'_q(z+S)=B'_q(z)$ is a periodic function of z, oscillating about zero average value, we note from Eq. (14) that the corresponding quadrupole focusing force $[\mathbf{F}'_{\rm foc}]_{\rm qf}$ in the beam frame is an oscillatory function of the argument $\gamma_b(z'+v_b t')(=z)$.

III. GLOBAL CONSERVATION CONSTRAINTS

In the remainder of this article, we specialize to the smooth-focusing model for the focusing force given by Eq. (12) in the beam frame. Here, we note from Eq. (12) that $[\mathbf{F}'_{\rm foc}]_{\rm sf} = -\nabla'\psi'_{\rm sf}(x',y')$, where the potential $\psi'_{\rm sf}(x',y')$ is defined by

$$\psi'_{\rm sf}(x',y') = \tfrac{1}{2}\gamma_b m\omega^2_{\beta\perp}(x'^2+y'^2). \tag{15}$$

The static potential $\psi'_{\rm sf}(x',y')$ in Eq. (15) provides transverse confinement of the beam particles in the $x'-y'$ plane, but not in the z' direction. That is, the charged particle beam is infinite and continuous in the z' direction for the choice of confining potential in Eq. (15). A simple generalization of Eq. (15) to the case of a finite-length charge bunch is to add to Eq. (15) a stationary $(\partial/\partial t'=0)$ contribution in the beam frame that provides axial confinement of the ions, e.g., a term proportional to $\gamma_b m\omega^2_{\beta z}z'^2/2$, where $\omega_{\beta z}=\text{const}$ is an effective betatron frequency for the axial motion. This gives the confining potential

$$\psi'_{\rm sf}(x',y',z') = \tfrac{1}{2}\gamma_b m\omega^2_{\beta\perp}(x'^2+y'^2) + \tfrac{1}{2}\gamma_b m\omega^2_{\beta z}z'^2. \tag{16}$$

Although Eq. (16) is stationary $(\partial/\partial t'=0)$ in the beam frame, when expressed in laboratory-frame variables we note from Eq. (4) that $z'^2=\gamma_b^2(z-v_b t)^2$. Therefore, the z' contribution in Eq. (16) corresponds to a potential well that travels in the z direction at the average axial velocity v_b of the beam particles. In this regard, it is convenient to view Eq. (16) as modeling the average potential of an rf bucket that provides a stationary confining potential centered at $z'=0$ in the beam frame. The relative axial and transverse dimensions of the charge bunch confined by Eq. (16) will of course depend on the ratio $\omega_{\beta z}/\omega_{\beta\perp}$ as well as the density of particles. In addition, as $\omega_{\beta z}\to 0$, Eq. (16) reduces to Eq. (15), as expected.

We now make use of the nonlinear Vlasov–Maxwell equations (5)–(8) with $\mathbf{F}'_{\rm foc} = -\nabla'\psi'_{\rm sf}(x',y',z')$ specified by Eq. (16) to derive certain global (spatially averaged) conservation constraints[10] in the beam frame that are useful in demonstrating a nonlinear stability theorem. First, it is convenient to rewrite Eq. (5) in the form of a continuity equation in the six-dimensional phase space $(\mathbf{x}',\mathbf{p}')$, i.e.,

$$\begin{aligned}& \frac{\partial f_b}{\partial t'} + \frac{\partial}{\partial \mathbf{x}'}\cdot(\mathbf{v}'f_b) \\ & + \frac{\partial}{\partial \mathbf{p}'}\cdot\left\{\left[Z_i e\left(\mathbf{E}' + \frac{1}{c}\mathbf{v}'\times\mathbf{B}'\right) - \nabla'\psi'_{\rm sf}\right]f_b\right\} = 0.\end{aligned} \tag{17}$$

Here, $\mathbf{p}'=m\mathbf{v}'$, and $\mathbf{B}'=\nabla'\times\mathbf{A}'$ and $\mathbf{E}'=\mathbf{E}'_L+\mathbf{E}'_T = -\nabla'\phi' - c^{-1}\partial\mathbf{A}'/\partial t'$ are determined self-consistently in terms of $f_b(\mathbf{x}',\mathbf{p}',t')$ from Maxwell's equations (7) and (8) with gauge condition $\nabla'\cdot\mathbf{A}'=0$ and boundary conditions given in Eq. (9). For present purposes, it is assumed that the phase-space density $f_b(\mathbf{x}',\mathbf{p}',t')$ is equal to zero beyond some radius r'_0, i.e., $f_b=0$ for $r'=(x'^2+y'^2)^{1/2}>r'_0<r_w$. For the case of a charge bunch with finite axial length, i.e., when $\omega_{\beta z}\neq 0$ in Eq. (16), it is also assumed that $f_b=0$ for $|z'|>L'_0/2$ where L'_0 is larger than the axial bunch length $2z'_b$.

One useful conservation relation[10] is the conservation of generalized entropy defined by $S'_G(t') = (1/L')\int d^3x'\int d^3p'\,G(f_b)$. Here $G(f_b)$ is a smooth differentiable function of f_b with $G(f_b\to 0)=0$. Substituting into Eq. (17) and integrating by parts with respect to $\mathbf{x}'$ and $\mathbf{p}'$ readily gives

$$\begin{aligned}\frac{d}{dt'}S'_G = {} & \frac{1}{L'}\int d^3x'\int d^3p'\,\frac{\partial G}{\partial f_b}\frac{\partial f_b}{\partial t'} \\ = {} & -\frac{1}{L'}\int d^3x'\int d^3p'\left(\frac{\partial}{\partial \mathbf{x}'}\cdot(\mathbf{v}'G)\right. \\ & \left. + \frac{\partial}{\partial \mathbf{p}'}\cdot\left\{\left[Z_i e\left(\mathbf{E}' + \frac{1}{c}\mathbf{v}'\times\mathbf{B}'\right) - \nabla'\psi'_{\rm sf}\right]G\right\}\right) = 0,\end{aligned} \tag{18}$$

where it is assumed that $f_b=0$ as $|\mathbf{p}'|\to\infty$. In Eq. (18), the domain of spatial integration is defined by $\int d^3x'/L'\cdots = \int_{-L'/2}^{L'/2}dz'/L'\int_0^{r_w}dr'\,r'\int_0^{2\pi}d\theta'\cdots$. Here, two cases are distinguished. Case (a) corresponds to an infinite-length beam where $\psi'_{\rm sf}(x',y')$ is specified by Eq. (15), and L' is viewed as a fundamental periodicity length for Fourier decomposition of the z' dependence of the distribution function and field components. Case (b) corresponds to a finite-length charge bunch, where $\psi'_{\rm sf}(x',y',z')$ is specified by Eq. (16) with $\omega_{\beta z}\neq 0$, and L' is chosen to be sufficiently large that $z'=L'/2$ and $z'=-L'/2$ are in the far-field regions of the charge bunch, where $\mathbf{E}'$ and $\mathbf{B}'$ are negligibly small. In either case, Eq. (18) gives the conservation relation

$$S_G'(t')=\frac{1}{L'}\int d^3x'\int d^3p'\,G(f_b)=\text{const}, \tag{19}$$

no matter how complicated the nonlinear evolution of f_b, $\mathbf{E}'$ and $\mathbf{B}'$. For the particular choices corresponding to $G(f_b)=f_b$ and $G(f_b)=-f_b\ln f_b$, Eq. (19) corresponds to the conservation of particle number and classical entropy, respectively.

A second global conservation relation of considerable practical importance is the conservation of the total internal energy $U'(t')$ carried by the particles and the fields.[10] Making use of the nonlinear Vlasov–Maxwell equations (17) and (6)–(8), some straightforward algebraic manipulation gives

$$\begin{aligned}\frac{d}{dt'}U'&=\frac{d}{dt'}\frac{1}{L'}\int d^3x'\\&\quad\times\left[\int d^3p'\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\text{sf}}'\right)f_b+\frac{\mathbf{E}'^2+\mathbf{B}'^2}{8\pi}\right]\\&=-\frac{1}{L'}\int d^3x'\,\nabla'\cdot\left(\frac{c}{4\pi}\,\mathbf{E}'\times\mathbf{B}'\right)=0.\end{aligned} \tag{20}$$

The Poyntang-flux contribution on the right-hand side of Eq. (20) vanishes when the divergence term is converted to a surface integral. This is because the tangential components of the electric field $\mathbf{E}'$ vanish at the perfectly conducting wall at $r'=r_w$ by virtue of the boundary conditions in Eq. (9), and because of the condition that the z' component of $\mathbf{E}'\times\mathbf{B}'$ is a periodic function of z' [case (a) above, for a continuous beam] or vanishes at $z'=\pm L'/2$ in the far-field region [case (b) above, for a finite-length charge bunch]. In both cases, Eq. (20) reduces to the global conservation constraint for the total particle plus field energy, i.e.,

$$\begin{aligned}U'(t')&=\frac{1}{L'}\int d^3x'\left[\int d^3p'\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\text{sf}}'\right)f_b+\frac{\mathbf{E}'^2+\mathbf{B}'^2}{8\pi}\right]\\&=\text{const}.\end{aligned} \tag{21}$$

Note that there are two contributions to the particle energy in Eq. (21). The term proportional to $(2m)^{-1}\mathbf{p}'^2f_b$ is associated with the particle kinetic energy in the beam frame, whereas the term proportional to $\psi_{\text{sf}}'f_b$ is the potential energy associated with the particle motion in the confining potential $\psi_{\text{sf}}'(x',y',z')$ of the applied focusing field.

Other global conservation constraints can also be derived from the nonlinear Vlasov–Maxwell equations (17) and (6)–(8) subject to the boundary conditions in Eq. (9) at $r'=r_w$. For example, it can be shown that the total particle plus field momentum[10] is conserved, i.e.,

$$\mathbf{P}'(t')=\frac{1}{L'}\int d^3x'\left(\int d^3p'\,\mathbf{p}'f_b+\frac{\mathbf{E}'\times\mathbf{B}'}{4\pi c}\right)=\text{const}. \tag{22}$$

Here, the term proportional to $\mathbf{p}'f_b$ represents the mechanical momentum carried by the particles, whereas the term proportional to $(4\pi c)^{-1}\mathbf{E}'\times\mathbf{B}'$ is the momentum carried by the fields.

IV. THREE-DIMENSIONAL KINETIC STABILITY THEOREM

A three-dimensional kinetic stability theorem can be derived by introducing a generalized Helmholtz free energy defined by $F(t')=U'(t')+S_G'(t')=\text{const}$, which is a linear combination of the entropy and energy conservation constraints in Eqs. (19) and (21). This gives

$$\begin{aligned}F'(t')&=\frac{1}{L'}\int d^3x'\Bigg\{\frac{|\mathbf{E}'|^2+|\mathbf{B}'|^2}{8\pi}\\&\quad+\int d^3p'\left[\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\text{sf}}'\right)f_b+G(f_b)\right]\Bigg\}\\&=\text{const}.\end{aligned} \tag{23}$$

Making use of $\mathbf{E}'=\mathbf{E}_L'+\mathbf{E}_T'$, where $\mathbf{E}_L'=-\nabla'\phi'$ and $\mathbf{E}_T'=-c^{-1}\partial\mathbf{A}'/\partial t'$, it is convenient to rewrite the electric field energy term in Eq. (23) as

$$\begin{aligned}\frac{1}{L'}\int d^3x'\,\frac{|\mathbf{E}'|^2}{8\pi}&=\frac{1}{L'}\int d^3x'\left(\frac{|\nabla'\phi'|^2}{8\pi}+2\nabla'\phi'\cdot\frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t'}+\frac{|\mathbf{E}_T'|^2}{8\pi}\right)\\&=\frac{1}{L'}\int d^3x'\left[\frac{|\nabla'\phi'|^2}{8\pi}+\frac{2}{c}\nabla'\cdot\left(\phi'\,\frac{\partial\mathbf{A}'}{\partial t'}\right)+\frac{|\mathbf{E}_T'|^2}{8\pi}\right]=\frac{1}{L'}\int d^3x'\left(\frac{|\nabla'\phi'|^2}{8\pi}+\frac{|\mathbf{E}_T'|^2}{8\pi}\right).\end{aligned} \tag{24}$$

Here, use has been made of $\nabla'\cdot\mathbf{A}'=0$, and the divergence term in Eq. (24) integrates to zero by virtue of the boundary condition $[\phi']_{r'=r_w}=0$ in Eq. (9) and the axial boundary conditions in cases (a) and (b) described above. Furthermore, it is useful to rewrite the electrostatic energy contribution in Eq. (24) in an alternate form that makes use of Poisson's equation (7) and the boundary condition on ϕ' in Eq. (9). From $|\nabla'\phi'|^2=\nabla'\cdot(\phi'\nabla'\phi')-\phi'\nabla'^2\phi'=\nabla'\cdot(\phi'\nabla'\phi')+4\pi Z_ie\phi'\int d^3p'f_b$, it is readily shown that

$$\frac{1}{L'}\int d^3x'\,\frac{|\nabla'\phi'|^2}{8\pi}=\frac{1}{L'}\int d^3x'\,\frac{1}{2}Z_ie\phi'\int d^3p'f_b. \tag{25}$$

Therefore, substituting Eqs. (24) and (25) into Eq. (23), it follows that $F'(t')$ can be expressed in the equivalent form

$$F'(t')=\frac{1}{L'}\int d^3x'\left\{\frac{|\mathbf{E}_T'|^2+|\mathbf{B}'|^2}{8\pi}+\int d^3p'\left[\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\rm sf}'+\frac{1}{2}Z_ie\phi'\right)f_b+G(f_b)\right]\right\}=\text{const}, \tag{26}$$

where $\mathbf{E}_T'=-c^{-1}\partial\mathbf{A}'/\partial t'$ and $\mathbf{B}'=\nabla'\times\mathbf{A}'$. Equation (26) is a powerful constraint condition on the nonlinear evolution of the system. Note that the representation in Eq. (26) naturally separates the electromagnetic field energy contribution with *transverse* polarization, i.e., the terms proportional to $(8\pi)^{-1}\int d^3x'\{|\mathbf{E}_T|^2+|\mathbf{B}'|^2\}$, from the field energy contribution in the *longitudinal* space-charge field, i.e., the term proportional to $\int d^3x'(Z_ie\phi'/2)\int d^3p' f_b$.

We now consider (arbitrary-amplitude) perturbations about a time-stationary $(\partial/\partial t'=0)$ equilibrium distribution $f_{\rm eq}(\mathbf{x}',\mathbf{p}')$ in the beam frame and corresponding space-charge potential $\phi_{\rm eq}'(\mathbf{x}')$ determined self-consistently from $\nabla'^2\phi_{\rm eq}'=-4\pi Z_ie\int d^3p' f_{\rm eq}$. It is further assumed that the equilibrium distribution $f_{\rm eq}(\mathbf{x}',\mathbf{p}')$ carries zero current in the beam frame, i.e., $\int d^3p'\mathbf{v}' f_{\rm eq}=0$, so that $\mathbf{A}_{\rm eq}'=0$ and $\mathbf{B}_{\rm eq}'=0=\mathbf{E}_{T_{\rm eq}}'$. Denoting perturbed quantities by $\delta f_b(\mathbf{x}',\mathbf{p}',t')=f_b(\mathbf{x}',\mathbf{p}',t')-f_{\rm eq}(\mathbf{x}',\mathbf{p}')$, $\delta\phi'(\mathbf{x}',t')=\phi'(\mathbf{x}',t')-\phi_{\rm eq}'(\mathbf{x}')$, $\delta\mathbf{E}_T'(\mathbf{x}',t)=\mathbf{E}_T'(\mathbf{x}',t')$ and $\delta\mathbf{B}'(\mathbf{x}',t')=\mathbf{B}'(\mathbf{x}',t')$, it follows from Eq. (26) that $\Delta F'(t')\equiv F'(t')-F_{\rm eq}'$ can be expressed as

$$\begin{aligned}\Delta F'(t')=\frac{1}{L'}\int d^3x'\Bigg\{&\frac{|\delta\mathbf{E}_T'|^2+|\delta\mathbf{B}'|^2}{8\pi}+\int d^3p'\Bigg[\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\rm sf}'\right)\delta f_b\\&+\frac{1}{2}Z_ie(\delta\phi'\delta f_b+\phi_{\rm eq}'\delta f_b+f_{\rm eq}\delta\phi')\\&+G(f_{\rm eq}+\delta f_b)-G(f_{\rm eq})\Bigg]\Bigg\}=\text{const}.\end{aligned} \tag{27}$$

Here, $\delta\phi'(\mathbf{x}',t')$ and $\phi_{\rm eq}'(\mathbf{x}')$ are related to $\delta f_b(\mathbf{x}',\mathbf{p}',t')$ and $f_{\rm eq}(\mathbf{x}',\mathbf{p}')$ by

$$\begin{aligned}&\nabla'^2\delta\phi'=-4\pi Z_ie\int d^3p'\,\delta f_b,\\&\nabla'^2\phi_{\rm eq}'=-4\pi Z_ie\int d^3p' f_{\rm eq},\end{aligned} \tag{28}$$

and $\psi_{\rm sf}'(x',y',z')$ is defined in Eq. (16) for an axially confined charge bunch, and in Eq. (15) for a continuous beam. Making use of Eq. (28), some straightforward algebra shows that $(Z_ie/2L')\int d^3x'\,\delta\phi'\int d^3p' f_{\rm eq}=(Z_ie/2L')\int d^3x'\,\phi_{\rm eq}'\int d^3p'\,\delta f_b$, and that $(Z_ie/2L')\int d^3x'\,\delta\phi'\int d^3p'\,\delta f_b=(1/L')\int d^3x'|\nabla'\delta\phi'|^2/8\pi$, where use is made of Eq. (9). Substituting into Eq. (27) then gives the equivalent constraint condition

$$\begin{aligned}\Delta F'(t')=\frac{1}{L'}\int d^3x'\Bigg\{&\frac{|\delta\mathbf{E}_T'|^2+|\delta\mathbf{B}'|^2+|\nabla'\delta\phi'|^2}{8\pi}\\&+\int d^3p'\Bigg[\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\rm sf}'+Z_ie\phi_{\rm eq}'\right)\delta f_b\\&+G(f_{\rm eq}+\delta f_b)-G(f_{\rm eq})\Bigg]\Bigg\}=\text{const}.\end{aligned} \tag{29}$$

The coefficient of $\delta f_b(\mathbf{x}',\mathbf{p}',t')$ in Eq. (29) will be recognized as the Hamiltonian

$$H'=\frac{1}{2m}\mathbf{p}'^2+\psi_{\rm sf}'(x',y',z')+\phi_{\rm eq}'(x',y',z') \tag{30}$$

for single-particle motion in the combined applied focusing potential $\psi_{\rm sf}'$ and equilibrium space-charge potential $\phi_{\rm eq}'$.

A *linear* (small-signal) stability theorem[10] can be obtained from Eq. (29) as follows. We Taylor expand $G(f_{\rm eq}+\delta f_b)=G(f_{\rm eq})+G'(f_{\rm eq})\delta f_b+G''(f_{\rm eq})(\delta f_b)^2/2+\cdots$, where $G'(f_{\rm eq})=\partial G(f_{\rm eq})/\partial f_{\rm eq}$, and retain terms to quadratic order in perturbed quantities. This gives

$$\begin{aligned}[\Delta F']^{(2)}=\frac{1}{L'}\int d^3x'\Bigg[&\frac{|\delta\mathbf{E}_T'|^2+|\delta\mathbf{B}'|^2+|\nabla'\delta\phi|^2}{8\pi}\\&+\int d^3p'\Bigg([H'+G'(f_{\rm eq})](\delta f_b)\\&+\frac{1}{2}G''(f_{\rm eq})(\delta f_b)^2+\cdots\Bigg)\Bigg]=\text{const}.\end{aligned} \tag{31}$$

We now choose $G(f_{\rm eq})$, which has been arbitrary to this point, to satisfy $\partial G(f_{\rm eq})/\partial f_{\rm eq}=-H'$ so that the term linear in δf_b vanishes exactly in Eq. (31). This condition also gives $G''(f_{\rm eq})=-\partial H'/\partial f_{\rm eq}$, so that Eq. (31) becomes (correct to second order in perturbed quantities)

$$[\Delta F']^{(2)}=\frac{1}{L'}\int d^3x'\left(\frac{|\delta\mathbf{E}_t'|^2+|\delta\mathbf{B}'|^2+|\nabla'\delta\phi|^2}{8\pi}+\frac{1}{2}\int d^3p'\frac{(\delta f_b)^2}{[-\partial f_{\rm eq}/\partial H']}\right)=\text{const}. \tag{32}$$

When $f_{\rm eq}(\mathbf{x}',\mathbf{p}')$ depends on $(\mathbf{x}',\mathbf{p}')$ only through the Hamiltonian H', and when $f_{\rm eq}(H')$ is a monotonically decreasing function of H' with

$$\frac{\partial}{\partial H'}f_{\rm eq}(H')\leqslant 0, \tag{33}$$

it follows that the quantity $[\Delta F']^{(2)}$ defined in Eq. (32) is a sum of positive-definite terms. Therefore, because $[\Delta F']^{(2)}=\text{const}$, no one of the terms in Eq. (32) can grow without bound, and we conclude that Eq. (33) is a *sufficient condition for linear stability* of the equilibrium $(f_{\rm eq},\phi_{\rm eq})$ to small-amplitude perturbations δf_b, $\delta\phi'$, $\delta\mathbf{E}_T'$, and $\delta\mathbf{B}'$.

The sufficient condition for stability in Eq. (33) is a very powerful result, applicable to a wide range of beam equilibria $f_{\rm eq}(H')$ that are spatially confined in the transverse $(x'-y')$ and axial (z') directions, and valid for longitudinal and transverse electromagnetic perturbations $\delta\phi',\delta\mathbf{E}_T',\delta\mathbf{B}'$ with arbitrary polarization. One example of a stable equilib-

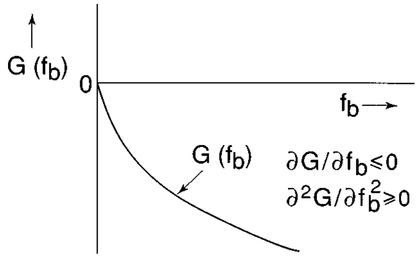

FIG. 1. Schematic of a possible functional form for $G(f_b)$ that satisfies the inequalities in Eq. (34).

rium is the thermal equilibrium distribution $f_{eq}=g(H')\equiv\beta'\exp(-H'/T_b')$, where β' and T_b' are positive constants.

The exact global constraint condition (29) can be used to show that Eq. (33) is also a sufficient condition for nonlinear stability of the equilibrium to perturbations with arbitrary amplitude. Proof of this *nonlinear stability theorem* makes two successive applications of the mean-value theorem and proceeds as follows. The functional form of $G(f_b)$ in Eq. (29) is quite general, subject only to the requirements that $G(f_b\to 0)=0$, that the integral $(1/L')\int d^3x'\int d^3p' G(f_b)$ converge, and that $G(f_b)$ be smooth and differentiable. In the subsequent proof of the nonlinear stability theorem, we exploit this generality and further assume that $G(f_b)$ is a monotonically decreasing function of f_b and has positive concavity, i.e.,

$$\frac{\partial}{\partial f_b}G(f_b)\le 0,$$
$$\frac{\partial^2}{\partial f_b^2}G(f_b)\ge 0, \tag{34}$$

over the entire range of values of the distribution function $f_b\ge 0$ accessible by the nonlinear Vlasov–Maxwell equations. Such a possible functional form for $G(f_b)$ is illustrated schematically in Fig. 1. Two successive applications of the mean-value theorem allows us to express the difference $G(f_{eq}+\delta f_b)-G(f_{eq})$ occurring in Eq. (29) in the form

$$\begin{aligned}G(f_{eq}+\delta f_b)-G(f_{eq})&=\frac{\partial G}{\partial f_b}\bigg|_{f_{eq}+\delta f_{b1}}\cdot\delta f_b\\&=\left(\frac{\partial G}{\partial f_b}\bigg|_{f_{eq}}+\frac{\partial^2 G}{\partial f_b^2}\bigg|_{f_{eq}+\delta f_{b2}}\delta f_{b1}\right)\delta f_b.\end{aligned}\tag{35}$$

Here, for positive perturbation $\delta f_b(\mathbf{x}',\mathbf{p}',t')\ge 0$, the quantities δf_{b1} and δf_{b2} lie in the intervals $0\le\delta f_{b2}\le\delta f_{b1}\le\delta f_b$, whereas for negative perturbation $\delta f_b\le 0$, the quantities δf_{b1} and δf_{b2} lie in the intervals $\delta f_b\le\delta f_{b1}\le\delta f_{b2}\le 0$. In either case, the product $\delta f_{b1}\delta f_b$ satisfies $\delta f_{b1}\delta f_b\ge 0$. Substituting Eq. (35) into Eq. (29) gives (exactly)

$$\begin{aligned}\Delta F'(t')=\frac{1}{L'}\int d^3x'\Bigg\{&\frac{|\delta\mathbf{E}_T'|^2+|\delta\mathbf{B}'|^2+|\nabla'\delta\phi'|^2}{8\pi}\\&+\int d^3p'\Bigg[\left(H'+\frac{\partial G}{\partial f_b}\bigg|_{f_{eq}}\right)\delta f_b\\&+\left(\frac{\partial^2 G}{\partial f_b^2}\bigg|_{f_{eq}+\delta f_{b2}}\right)(\delta f_{b1}\delta f_b)\Bigg]\Bigg\}=\text{const.}\end{aligned}\tag{36}$$

As before, we eliminate the term linear in δf_b in Eq. (36) by choosing $[\partial G/\partial f_b]_{f_{eq}}=-H'$, which also implies that $[\partial^2 G/\partial f_b^2]_{f_{eq}}=-\partial H'/\partial f_{eq}$. Equation (36) then reduces to

$$\begin{aligned}\Delta F'(t')=\frac{1}{L'}\int d^3x'\Bigg\{&\frac{|\delta\mathbf{E}_T'|^2+|\delta\mathbf{B}'|^2+|\nabla'\delta\phi'|^2}{8\pi}\\&+\int d^3p'\left(\frac{\partial^2 G}{\partial f_b^2}\bigg|_{f_{eq}+\delta f_{b2}}\right)(\delta f_{b1}\delta f_b)\Bigg\}=\text{const.}\end{aligned}\tag{37}$$

Because of the assumption $\partial^2 G/\partial f_b^2\ge 0$ in Eq. (34), and because $\delta f_{b1}\delta f_b\ge 0$ follows by construction from the mean-value theorem, we conclude that the right-hand side of Eq. (37) is a sum of positive-definite terms, no one of which can grow without bound. Therefore, because $[\partial^2 G/\partial f_b^2]_{f_{eq}}=-\partial H'/\partial f_{eq}\ge 0$, by assumption, we conclude that $\partial f_{eq}(H')/\partial H'\le 0$ is a sufficient condition for nonlinear stability.

V. THERMAL EQUILIBRIUM—A STATE OF MINIMUM HELMHOLTZ FREE ENERGY

There is clearly a wide range of choices of distribution functions $f_{eq}(H')$ for which $\partial f_{eq}/\partial H'\le 0$ and the equilibrium is therefore stable. As noted earlier, one such distribution is the thermal equilibrium distribution $f_{eq}=g(H')\equiv\beta'\exp(-H'/T_b')$, where β' and T_b' are positive constants. Thermal equilibrium properties of nonneutral plasmas have been widely studied for a cylindrical plasma column confined radially by a uniform axial magnetic field,[1,14,17] and for a non-neutral plasma confined radially and axially in a Malmberg–Penning trap.[18,19] For completeness, before summarizing thermal equilibrium properties of an intense nonneutral charge bunch[20] in the present application, we present a short proof that demonstrates that the thermal equilibrium distribution $g(H')=\beta'\exp(-H'/T_b')$ is that unique choice of distribution function that *minimizes* the classical Helmholtz free energy $F_H'(t')=U'(t')-T_b'S'(t')-\mu_b'N'(t')=\text{const.}$[13] Here, $U(t')$ is the internal energy defined in Eq. (21), $S'=-(1/L')\int d^3x'd^3p' f_b\ln f_b$ is the classical entropy, $N'=(1/L')\int d^3x'd^3p' f_b$ is the number of particles per unit axial length, and $\mu_b'=T_b'(1+\ln\beta')=\text{const}$ is the chemical potential. This corresponds to choosing the entropy function to be $G(f_b)=T_b'f_b\ln(f_b/\beta')-T_b'f_b$ in Eq. (26), which gives

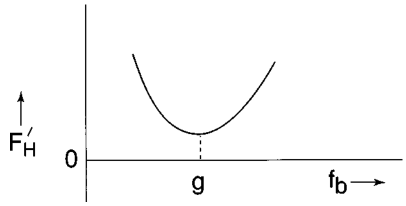

FIG. 2. The thermal equilibrium distribution function $f_b = g(H')\equiv\beta'\exp(-H'/T_b')$ minimizes the classical Helmholtz free energy F_H' defined in Eq. (38).

$$F_H'(t') = \frac{1}{L'}\int d^3x'\left\{\frac{|\mathbf{E}_T'|^2+|\mathbf{B}'|^2}{8\pi} + \int d^3p'\left[\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\rm sf}'+\frac{1}{2}Z_ie\phi'\right)f_b + T_b'f_b\ln(f_b/\beta') - T_b'f_b\right]\right\} = \text{const.} \tag{38}$$

Equation (38) is an exactly conserved quantity no matter how complicated the nonlinear evolution of $f_b(\mathbf{x}',\mathbf{p}',t')$, $\mathbf{E}_T'(\mathbf{x}',t')$, $\mathbf{B}'(\mathbf{x}',t')$ and $\phi'(\mathbf{x},t')$ according to the Vlasov–Maxwell equations (7), (8) and (17).

We now pose the question: what is the choice of distribution function $f_b = g$ that minimizes the Helmholtz free energy F_H' defined in Eq. (38). That is, what is the choice of distribution function for which

$$[\delta F_H']_{f_b=g} = 0, \qquad \{\delta[\delta F_H']\}_{f_b=g} \geq 0, \tag{39}$$

where $\delta(\cdots)$ denotes variation with respect to f_b. Taking the first variation of Eq. (38) gives

$$\delta F_H' = \frac{1}{L'}\int d^3x'\left\{\frac{\mathbf{E}_T'\cdot\delta\mathbf{E}_T'+\mathbf{B}'\cdot\delta\mathbf{B}'}{4\pi} + \int d^3p'\left[\left(\frac{\mathbf{p}'^2}{2m}+\psi_{\rm sf}'+Z_ie\phi'\right) + T_b'\ln(f_b/\beta')\right]\delta f_b\right\} = 0. \tag{40}$$

Here, use has been made of $\int d^3x'd^3p'\frac{1}{2}Z_ie\,\delta(\phi'f_b) = \int d^3x'd^3p'(Z_ie/2)(\phi'\delta f_b + f_b\delta\phi') = \int d^3x'd^3p'\,\phi'\,\delta f_b$. [See also discussion following Eqs. (27) and (28).] Setting the coefficient of δf_b equal to zero in Eq. (40) gives directly

$$f_b = g(H') \equiv \beta'\exp(-H'/T_b'), \tag{41}$$

where $H' = \mathbf{p}'^2/2m + \psi_{\rm sf}' + Z_ie\phi_g'$ is the single-particle Hamiltonian, and the electrostatic potential ϕ_g' is determined self-consistently from Poisson's equation

$$\nabla'^2\phi_g' = -4\pi Z_ie\int d^3p'\,g(H'). \tag{42}$$

Because the distribution function $g(H')$ carries zero current in the beam frame, i.e., $\int d^3p'\mathbf{v}'g(H')=0$, it follows from Maxwell's equations (7) and (8) that $[\mathbf{A}']_g=0$ and $[\mathbf{E}_T']_g = 0 = [\mathbf{B}']_g$. Therefore, the first two terms on the right-hand side of Eq. (40) also vanish when $f_b = g(H')$. Taking the second variation of Eq. (38), and evaluating for $f_b = g$, it is readily shown that

$$\{\delta[\delta F_H']\}_{f_b=g} = \frac{1}{L'}\int d^3x'\left(\frac{|\delta\mathbf{E}_T'|^2+|\delta\mathbf{B}'|^2+|\nabla'\delta\phi'|^2}{4\pi} + T_b'\int d^3p'\,\frac{(\delta f_b)^2}{g}\right) \geq 0, \tag{43}$$

where $T_b'>0$ is assumed. We therefore conclude that not only is the thermal equilibrium distribution defined in Eq. (41) nonlinearly stable (Sec. IV), it is the unique choice of distribution function that *minimizes* the classical Helmholtz free energy F_H' [Eqs. (39), (40) and (43) and Fig. 2]. Conservation of the classical Helmholtz free energy, $F_H'(t') = \text{const}$, also provides a very powerful constraint condition that can be used to estimate nonlinear bounds on the unstable field energy $\Delta\mathscr{E}_F'(t')$ that can develop from initial (unstable) distribution functions $f_b(\mathbf{x}',\mathbf{p}',t'=0)$. This has proved to be a useful technique in applications to uniform neutral plasmas[13,21] and to nonrelativistic nonneutral plasmas.[22] It's application to intense nonneutral beams will be the subject of a future investigation.

We now return to a brief examination of Eqs. (41) and (42). Assuming axisymmetric space-charge potential $\phi_g'(r',z')$, where $r' = (x'^2+y'^2)^{1/2}$, the thermal equilibrium distribution in Eq. (41) can also be expressed in beam-frame variables as

$$g(H') = \frac{\hat{n}_b'\exp(Z_ie\hat{\phi}_g'/T_b')}{(2\pi mT_b')^{3/2}} \times\exp\left[-\frac{1}{T_b'}\left(\frac{\mathbf{p}'^2}{2m}+Z_ie\phi_g'(r',z') + \frac{1}{2}\gamma_bm\omega_{\beta\perp}^2r'^2 + \frac{1}{2}\gamma_bm\omega_{\beta z}^2z'^2\right)\right]. \tag{44}$$

Here, $\hat{n}_b'$ and $\hat{\phi}_g'$ are constants identified with the values of the space-charge potential and particle density, respectively, at the center of the charge bunch, i.e., $\hat{n}_b' \equiv n_g'(r'=0, z'=0)$ and $\hat{\phi}_g' \equiv \phi_g'(r'=0, z'=0)$. Evaluating the particle density $n_g'(r',z') = \int d^3p'\,g(H')$ from Eq. (44) gives

$$n_g'(r',z') = \hat{n}_b'\exp\left(-\frac{Z_ie}{T_b'}[\phi_g'(r',z')-\hat{\phi}_g'] - \frac{\gamma_bm}{2T_b'}(\omega_{\beta\perp}^2r'^2+\omega_{\beta z}^2z'^2)\right). \tag{45}$$

The equilibrium space-charge potential $\phi_g'(r',z')$ occurring in the exponent in Eq. (45) must of course be determined self-consistently from Poisson's equation (42), which becomes

$$\left(\frac{1}{r'}\frac{\partial}{\partial r'}r'\frac{\partial}{\partial r'}+\frac{\partial^2}{\partial z'^2}\right)\phi_g'(r',z')$$
$$=-4\pi Z_i e\hat{n}_b'\exp\left(-\frac{Z_i e}{T_b'}[\phi_g'(r',z')-\hat{\phi}_g']-\frac{\gamma_b m}{2T_b'}(\omega_{\beta\perp}^2 r'^2+\omega_{\beta z}^2 z'^2)\right). \quad (46)$$

The (nonlinear) Poisson equation (46) can be solved numerically for $\phi_g'(r',z')$ subject to the boundary condition $\phi_g'(r'=r_w,z')=0$ at the conducting wall [Eq. (9)]. The detailed solution to Eq. (46) of course depends on the thermal emittance (proportional to T_b'), proximity of the conducting wall at $r'=r_w$, and the relative values of $\omega_{\beta\perp}^2$, $\omega_{\beta z}^2$ and $\hat{\omega}_{pb}'^2=4\pi\hat{n}_b' Z_i^2 e^2/m$. Here, $\hat{\omega}_{pb}'^2$ is the nonrelativistic plasma frequency-squared in the beam frame at $(r',z')=(0,0)$, which is a measure of the strength of (defocusing) space-charge effects.

Detailed numerical solutions to Eq. (46) will not be presented here. Rather, for purposes of illustration, we consider the simple limiting case of very low thermal emittance ($T_b'\to 0$), and assume that the conducting wall is far removed from the charge bunch ($r_w \gg r_b'$). In this case, the beam density is approximately uniform with $n_g'(r',z')\simeq\hat{n}_b'=$const inside a spheroidal region defined by $0\leqslant r'^2/r_b'^2+z'^2/z_b'^2<1$, and equal to zero for $r'^2/r_b'^2+z'^2/z_b'^2>1$. Inside the spheroid, it follows from Eq. (45) for $T_b'\to 0$ that $\phi_g'(r',z')-\hat{\phi}_g'$ is given approximately by

$$Z_i e(\phi_g'-\hat{\phi}_g')=-\tfrac{1}{2}\gamma_b m(\omega_{\beta\perp}^2 r'^2+\omega_{\beta z}^2 z'^2) \quad (47)$$

for $r'^2/r_b'^2+z'^2/z_b'^2<1$. Substituting Eq. (47) into Eq. (46) and evaluating inside the spheroid then gives the condition

$$\frac{1}{2\gamma_b}\hat{\omega}_{pb}'^2=\omega_{\beta\perp}^2+\frac{1}{2}\omega_{\beta z}^2, \quad (48)$$

which relates $\hat{\omega}_{pb}'^2$, $\omega_{\beta\perp}^2$ and $\omega_{\beta z}^2$. For specified focusing frequencies $\omega_{\beta\perp}$ and $\omega_{\beta z}$, Eq. (48) should be viewed as determining the limiting space-charge density ($\hat{\omega}_{pb}'^2$) that can be confined in the limit $T_b'\to 0$. Equation (47) should be compared with the electrostatic potential calculated from Poisson's equation inside an isolated, uniformly charged spheroid with uniform charge density $Z_i e\hat{n}_b'$ in the region $r'^2/r_b'^2+z'^2/z_b^2<1$. The corresponding potential inside the spheroid is found to be[1,16]

$$Z_i e(\phi_{eq}'-\hat{\phi}_{eq}')=-\tfrac{1}{4}m\hat{\omega}_{pb}'^2(\alpha r'^2+\beta z'^2), \quad (49)$$

where $\alpha+\beta/2=1$, and the values of the constants α and β depend on r_b'/z_b' and whether or not the spheroid is *elongated* ($z_b'>r_b'$) or *oblate* ($z_b'<r_b'$). For example, for an elongated spheroid with $z_b'>r_b'$, the constants α and β are defined by[1,16]

$$\begin{aligned}\alpha&=\frac{1}{(1-r_b'^2/z_b'^2)}-\frac{r_b'^2/z_b'^2}{2(1-r_b'^2/z_b'^2)^{3/2}}\times\ln\left|\frac{1+(1-r_b'^2/z_b'^2)^{1/2}}{1-(1-r_b'^2/z_b'^2)^{1/2}}\right|,\\ \frac{1}{2}\beta&=-\frac{r_b'^2/z_b'^2}{(1-r_b'^2/z_b'^2)}+\frac{r_b'^2/z_b'^2}{2(1-r_b'^2/z_b'^2)^{3/2}}\times\ln\left|\frac{1+(1-r_b'^2/z_b'^2)^{1/2}}{1-(1-r_b'^2/z_b'^2)^{1/2}}\right|,\end{aligned} \quad (50)$$

where $\alpha+\beta/2=1$. Finally, comparing the coefficients of r'^2 and z'^2 in Eqs. (47) and (49) gives

$$\begin{aligned}\alpha&=\frac{2\gamma_b\omega_{\beta\perp}^2}{\hat{\omega}_{pb}'^2}=\frac{2\omega_{\beta\perp}^2}{2\omega_{\beta\perp}^2+\omega_{\beta z}^2},\\ \frac{1}{2}\beta&=\frac{\gamma_b\omega_{\beta z}^2}{\hat{\omega}_{pb}'^2}=\frac{\omega_{\beta z}^2}{2\omega_{\beta\perp}^2+\omega_{\beta z}^2},\end{aligned} \quad (51)$$

where use has been made of Eq. (48). For weak axial focusing force with $\omega_{\beta z}^2\ll\omega_{\beta\perp}^2$, it follows from Eq. (51) that $\beta\simeq\omega_{\beta z}^2/\omega_{\beta\perp}^2\ll 1$ and $\alpha\simeq 1-\omega_{\beta z}^2/2\omega_{\beta\perp}^2$. In this case, it follows from Eq. (50) that the charge bunch is highly elongated with $z_b'\gg r_b'$. On the other hand, for $\omega_{\beta z}^2=\omega_{\beta\perp}^2=\hat{\omega}_{pb}'^2/3\gamma_b$, Eqs. (50) and (51) give $\alpha=\beta=2/3$ and $z_b'=r_b'$ corresponding to a *spherical* charge bunch. For an oblate spheroid ($z_b'<r_b'$), alternate expressions[1] for α and β to those in Eq. (50) must be used.

The previous analysis summarizes thermal equilibrium properties for a very low-emittance, space-charge-dominated charge bunch. Analysis of Eqs. (41) and (42) also simplifies for a high-emittance, low-intensity beam in which $Z_i e|\phi_g'-\hat{\phi}_g'|\ll\psi_{sf}'$ and $\hat{\omega}_{pb}'^2/2\gamma_b\ll\omega_{\beta\perp}^2+\omega_{\beta z}^2/2$. In this case, Eq. (45) reduces in lowest order to

$$n_g'(r',z')=\hat{n}_b'\exp\left\{-\left(\frac{r'^2}{r_b'^2}+\frac{z'^2}{z_b'^2}\right)\right\}, \quad (52)$$

where $r_b'^2\equiv 2T_b'/\gamma_b m\omega_{\beta\perp}^2$, $z_b'^2\equiv 2T_b'/\gamma_b m\omega_{\beta z}^2$, and $z_b'^2/r_b'^2=\omega_{\beta\perp}^2/\omega_{\beta z}^2$. While the constant-density contours in Eq. (52) are also spheroidal in shape, it is clear from Eq. (52) that the density profile is *diffuse* in the emittance-dominated regime, rather than *uniform* within a sharp spheroidal boundary as in the low-emittance, space-charge-dominated case.

VI. CONCLUSIONS

In the present analysis, we Lorentz-transformed the nonlinear Vlasov–Maxwell equations to the beam frame where the particle motion is nonrelativistic (Sec. II), and made use of global (spatially averaged) conservation constraints (Sec. III) to derive a three-dimensional kinetic stability theorem (Sec. IV). The analysis was carried out for the case where the applied focusing potential in the beam frame (assumed time-stationary) is of the form $\psi_{sf}'(\mathbf{x}')=(\gamma_b m/2)[\omega_{\beta\perp}^2(x'^2+y'^2)+\omega_{\beta z}^2 z'^2]$, where $\omega_{\beta\perp}$ and $\omega_{\beta z}$ are constant focusing frequencies. It was shown that a *sufficient condition for linear and nonlinear stability* for perturbations about a beam equilibrium distribution $f_{eq}(\mathbf{x}',\mathbf{p}')$ is that $\partial f_{eq}(H')/\partial H'\leqslant 0$ [Eq.

(33)], where $H'=\mathbf{p}'^2/2m+\psi'_{\mathrm{sf}}(\mathbf{x}')+\phi'_{\mathrm{eq}}(\mathbf{x}')$ is the single-particle energy defined in Eq. (30), and $\phi'_{\mathrm{eq}}(\mathbf{x}')$ is the equilibrium space-charge potential. It was also shown (Sec. V) that the thermal equilibrium distribution $f_{\mathrm{eq}}=g(H')\equiv\beta'\exp(-H'/T'_b)$, where β' and T'_b are positive constants, is the unique choice of distribution function that minimizes the classical Helmholtz free energy $F'_H(t')=\mathrm{const}$ defined in Eq. (38). Thermal equilibrium properties of a charge bunch were summarized (Sec. V) for the two limiting regimes corresponding to a low-emittance, high-intensity beam with strong space-charge effects, and a low-intensity, high-emittance beam with weak space-charge effects.

It is very important to recognize the wide range of applicability of the three-dimensional stability theorem developed in the present analysis. Most notably, it applies to perturbations about equilibria $f_{\mathrm{eq}}(H')$ with arbitrary polarization and initial amplitude; to *continuous beams* that are radially confined and infinite in axial extent ($\omega_{\beta\perp}\neq 0,\omega_{\beta z}=0$); to *charge bunches* that are radially and axially confined ($\omega_{\beta\perp}\neq 0$ and $\omega_{\beta z}\neq 0$); and to beams with arbitrary space-charge intensity consistent with the requirement that the applied focusing potential $\psi'_{\mathrm{sf}}(\mathbf{x}')$ provide confinement of the beam particles. As a final point, it should be emphasized that the stability theorem developed here has far wider applicability than to the case where $\psi'_{\mathrm{sf}}(\mathbf{x}')$ has the simple quadratic dependence on x', y' and z' in Eq. (16), provided the confining potential is time-stationary in the beam frame, i.e., $\partial\psi'_{\mathrm{sf}}/\partial t'=0$. The main requirement on the $\mathbf{x}'$ dependence is that $\psi'_{\mathrm{sf}}(\mathbf{x}')$ correspond to a *confining* potential, i.e., that the focusing force $[\mathbf{F}'_{\mathrm{foc}}]_{\mathrm{sf}}=-\nabla'\psi'_{\mathrm{sf}}$ is restoring.

Important future generalizations of the present analysis will include: (a) use of the global conservation constraint corresponding to the classical Helmholtz free energy $F'_H(t')=\mathrm{const}$ defined in Eq. (38) to determine nonlinear bound estimates on the unstable field energy $\Delta\mathscr{E}'_F(t')$ that can be released for various choices of (unstable) initial distribution function $f_b(\mathbf{x}',\mathbf{p}',t'=0)$; and (b) investigation of stability properties for the case of a periodic-focusing quadrupole lattice where the focusing force in the beam frame depends on t' [see Eq. (14)].

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[1] R. C. Davidson, *Physics of Nonneutral Plasmas* (Addison-Wesley, Reading, MA, 1990), and references therein.
[2] T. P. Wangler, *Principles of RF Linear Accelerators* (Wiley, New York, 1998).
[3] M. Reiser, *Theory and Design of Charged Particle Beams* (Wiley, New York, 1994).
[4] R. W. Müller, in *Nuclear Fusion by Inertial Confinement: A Comprehensive Treatise*, edited by G. Velarde, Y. Ronen, and J. M. Martinez-Val (Chemical Rubber Co., Boca Raton, FL, 1993), Chap. 17, pp. 437–453.
[5] E. P. Lee and J. Hovingh, Fusion Technol. **15**, 369 (1989).
[6] R. A. Jameson, in *Advanced Accelerator Concepts*, edited by J. S. Wurtele, American Institute of Physics Conference Proceedings 279 (American Institute of Physics, New York, 1993), p. 969.
[7] R. C. Davidson and C. Chen, ''Kinetic Description of High-Intensity Beam Propagation Through a Periodic Focusing Field,'' Particle Accelerators (to be published), and references therein.
[8] J. Struckmeier and I. Hofmann, Part. Accel. **39**, 219 (1992).
[9] I. Hofmann, L. Laslett, L. Smith, and I. Haber, Part. Accel. **13**, 145 (1983).
[10] See, for example, in Ref. 1 Chapter 2 and pages 130–137, and in Chapter 4, and references therein.
[11] W. Newcomb, as reported by I. B. Bernstein, Phys. Rev. **109**, 10 (1958).
[12] C. S. Gardner, Phys. Fluids **6**, 839 (1963).
[13] T. K. Fowler, J. Math. Phys. **4**, 559 (1963).
[14] R. C. Davidson and N. A. Krall, Phys. Fluids **13**, 1543 (1970).
[15] S. M. Lund and R. C. Davidson, Phys. Plasmas **5**, 3028 (1998).
[16] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1971).
[17] R. C. Davidson and S. M. Lund, *American Institute of Physics Conference Proceedings on Advances in Plasma Physics* (American Institute of Physics, Woodbury, NY, 1994), Vol. 314, p. 1.
[18] S. A. Prasad and T. M. O'Neil, Phys. Fluids **22**, 278 (1979).
[19] D. H. E. Dubin and T. M. O'Neil, ''Trapped Nonneutral Plasmas, Liquids and Crystals—The Thermal Equilibrium State,'' Rev. Mod. Phys. (to be published).
[20] N. Brown and M. Reiser, Phys. Plasmas **2**, 965 (1995).
[21] R. C. Davidson and S. T. Tsai, J. Plasma Phys. **9**, 101 (1973).
[22] R. C. Davidson, J. Plasma Phys. **5**, 157 (1985).