

Intense Sheet Beam Stability Properties for Uniform Phase-Space Density*

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Abstract

A self-consistent one-dimensional waterbag equilibrium $f_b^0(x, p_x)$ for a sheet beam propagating through a smooth focusing field is shown to be exactly solvable for the beam density $n_b^0(x)$ and space-charge potential $\phi^0(x)$. A closed Schrodinger-like eigenvalue equation is derived for small-amplitude perturbations, and the WKB approximation is employed to determine the eigenfrequency spectrum as a function of the normalized beam intensity $s_b = \hat{\omega}_{pb}^2 / \gamma_b^2 \omega_{\beta\perp}^2$, where $\hat{\omega}_{pb}^2 = 4\pi\hat{n}_b e_b^2 / \gamma_b m_b$ is the relativistic plasma frequency-squared and $\hat{n}_b = n_b(x=0)$ is the on-axis number density of beam particles.

SHEET BEAM EQUILIBRIUM WITH UNIFORM PHASE-SPACE DENSITY

We consider an intense sheet beam [1], made up of particles with charge e_b and rest mass m_b , which propagates in the z -direction with directed kinetic energy $(\gamma_b - 1)m_b c^2$ and average axial velocity $V_b = \beta_b c = \text{const}$. Here, $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor, c is the speed of light *in vacuo*, and the beam is assumed to be uniform in the y - and z - directions with $\partial/\partial y = 0 = \partial/\partial z$. The beam is centered in the x - direction at $x = 0$, and transverse confinement is provided by an applied focusing force, $F_x^{foc} = -\gamma_b m_b \omega_{\beta\perp}^2 x$, with $\omega_{\beta\perp}^2 = \text{const}$ in the smooth focusing approximation. The transverse dimension of the sheet beam is denoted by $2x_b$, and planar, perfectly conducting walls are located at $x = \pm x_w$. The particle motion in the beam frame is assumed to be nonrelativistic, and we introduce the effective potential $\psi(x, t)$ defined by

$$\psi(x, t) = \frac{1}{2}\gamma_b m_b \omega_{\beta\perp}^2 x^2 + \frac{1}{\gamma_b^2} e_b \phi(x, t). \quad (1)$$

The Vlasov-Maxwell equations describing the self-consistent nonlinear evolution of $f_b(x, p_x, t)$ and $\psi(x, t)$ can be expressed as [2]

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial p_x} \right) f_b = 0, \quad (2)$$

and

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma_b m_b \omega_{\beta\perp}^2 - \frac{4\pi e_b^2}{\gamma_b^2} \int_{-\infty}^{\infty} dp_x f_b. \quad (3)$$

As an equilibrium example ($\partial/\partial t = 0$) that is analytically tractable, we consider the choice of distribution function

$$F_b(H_\perp) = \frac{\hat{n}_b}{(8\gamma_b m_b \hat{H}_\perp)^{1/2}} \Theta(H_\perp - \hat{H}_\perp), \quad (4)$$

where $H_\perp = p_x^2/2\gamma_b m_b + \psi^0(x)$ is the transverse Hamiltonian, $\Theta(x)$ is the Heaviside step-function, and \hat{n}_b, \hat{H}_\perp are positive constants. Evaluating the number density $n_b^0(x) = \int_{-\infty}^{\infty} dp_x F_b(H_\perp)$, we readily obtain

$$n_b^0(x) = \begin{cases} \hat{n}_b \left[1 - \psi^0(x)/\hat{H}_\perp \right]^{1/2}, & -x_b < x < x_b, \\ 0, & |x| > x_b. \end{cases} \quad (5)$$

Here, the location of the beam edge ($x = \pm x_b$) is determined from

$$\psi^0(x = \pm x_b) = \hat{H}_\perp, \quad (6)$$

where $\psi^0(x=0) = 0$ is assumed. It is useful to introduce the effective Debye length λ_D defined by

$$\lambda_D^2 = \frac{\gamma_b^3 \hat{H}_\perp}{4\pi \hat{n}_b e_b^2} = \frac{1}{2} \frac{\gamma_b^2 \hat{v}_0^2}{\hat{\omega}_{pb}^2}. \quad (7)$$

Here, $\hat{v}_0 = (2\hat{H}_\perp/\gamma_b m_b)^{1/2}$ is the maximum speed of a particle with energy \hat{H}_\perp as it passes through $x = 0$. Substituting Eq. (5) into Eq. (3) then gives

$$\frac{\partial^2}{\partial x^2} \left(\frac{\psi^0(x)}{\hat{H}_\perp} \right) = \frac{1}{\lambda_D^2} \left(\frac{1}{s_b} - \left[1 - \frac{\psi^0(x)}{\hat{H}_\perp} \right]^{1/2} \right) \quad (8)$$

in the beam interior ($-x_b < x < x_b$). Equation (8) is to be integrated subject to the boundary conditions $[\psi^0]_{x=0} = 0 = [\partial\psi^0/\partial x]_{x=0}$. For physically acceptable solutions to Eq. (8), the condition $[\partial^2\psi^0/\partial x^2]_{x=0} > 0$ imposes the requirement that s_b lies in the interval $0 < s_b < 1$, where $s_b = \hat{\omega}_{pb}^2/\gamma_b^2 \omega_{\beta\perp}^2$. The regime $s_b \ll 1$ corresponds to a low-intensity, emittance-dominated beam, whereas the regime $s_b \rightarrow 1$ corresponds to a low-emittance, space-charge-dominated beam. In solving Eq. (8), it is convenient to introduce the dimensionless variables defined by

$$X = \frac{x}{\lambda_D}, \quad \hat{\psi}^0(X) = \frac{\psi^0(x)}{\hat{H}_\perp}. \quad (9)$$

Substituting Eq. (9) into Eq. (8), integrating once, and enforcing $[\psi^0]_{x=0} = 0 = [\partial\psi^0/\partial x]_{x=0}$, gives

$$\frac{1}{2} \left(\frac{d\hat{\psi}^0}{dX} \right)^2 = \frac{1}{s_b} \hat{\psi}^0 + \frac{2}{3} \left[(1 - \hat{\psi}^0)^{3/2} - 1 \right] \quad (10)$$

in the interval $-x_b/\lambda_D \leq X \leq x_b/\lambda_D$. Equation (10) can be integrated exactly to determine X as a function of $(1 - \hat{\psi}^0)^{1/2} = n_b^0(X)/\hat{n}_b$ [see Eq. (5)]. We express $X =$

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$\int_0^{\widehat{\psi}^0} d\widehat{\psi}^0 / (d\widehat{\psi}^0/dX)$, change variables to $z = (1 - \widehat{\psi}^0)^{1/2}$, and make use of Eq. (10). This gives [1, 3]

$$X = 3^{1/2} \int_{(1-\widehat{\psi}^0)^{1/2}}^1 \frac{z dz}{[(1-z)(a^+ - z)(z - a^-)]^{1/2}}, \quad (11)$$

where a^+ and a^- are defined by

$$a^\pm = \frac{1}{4s_b} \{3 - 2s_b \pm [3(3 + 4s_b - 4s_b^2)]^{1/2}\}. \quad (12)$$

From Eqs. (6) and (11) we obtain a closed expression for x_b/λ_D in terms of the normalized beam intensity s_b for the choice of equilibrium distribution function in Eq. (4). The areal density of the beam particles, $N_b = \int_{-x_b}^{x_b} dx n_b^0(x)$, for the density profile in Eq. (5) can be expressed as

$$N_b = 2\widehat{n}_b \int_0^{x_b} dx [1 - \psi^0(x)/\widehat{H}_\perp]^{1/2}. \quad (13)$$

Some algebraical manipulation that make use of Eqs. (9), (10) and (13) gives

$$\frac{N_b}{2\widehat{n}_b x_b} = 3^{1/2} \frac{\lambda_D}{x_b} \int_0^1 \frac{z^2 dz}{[(1-z)(a^+ - z)(z - a^-)]^{1/2}}, \quad (14)$$

where x_b/λ_D is determined from Eq. (11). Note that $N_b/2\widehat{n}_b x_b$ depends only on the dimensionless intensity parameter s_b . Typical normalized density profiles

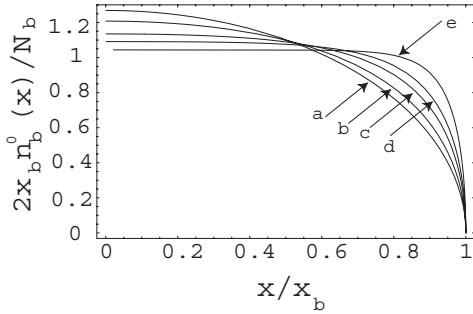


Figure 1: Plots of the normalized density profile $2x_b n_b^0(x)/N_b$ versus x/x_b for different values of the normalized beam intensity s_b corresponding to (a) $s_b = 0.2$, (b) $s_b = 0.9$, (c) $s_b = 0.99$, (d) $s_b = 0.999$, (e) $s_b = 0.999999$.

$2x_b n_b^0(x)/N_b$ are illustrated in Fig.1 for values of s_b ranging from $s_b = 0.2$ to $s_b = 0.999999$ [1]. Finally, defining the equilibrium transverse pressure profile by $P_b^0(x) = \int_{-\infty}^{\infty} dp_x (p_x^2/\gamma_b m_b) f_b^0$, we readily obtain

$$P_b^0(x) = \frac{4}{3} \widehat{n}_b \widehat{H}_\perp \left[1 - \frac{\psi^0(x)}{\widehat{H}_\perp} \right]^{3/2}. \quad (15)$$

Comparing Eqs. (5) and (15), note that $P_b^0(x) = \text{const}[n_b^0(x)]^3$, which corresponds to a triple-adiabatic pressure relation.

LINEARIZED EQUATIONS AND STABILITY ANALYSIS

The *linearized* Vlasov-Maxwell equations can be expressed as [2]

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} - \frac{\partial \psi^0}{\partial x} \frac{\partial}{\partial p_x} \right) \delta f_b = v_x \frac{\partial \delta \psi}{\partial x} \frac{\partial F_b}{\partial H_\perp}, \quad (16)$$

and

$$\frac{\partial^2}{\partial x^2} \delta \psi = -\frac{4\pi e_b^2}{\gamma_b^2} \delta n_b, \quad (17)$$

where $\delta n_b(x, t) = \int_{-\infty}^{\infty} dp_x \delta f_b$ is the perturbed number density of beam particles. In analyzing Eqs. (16) and (17), it is convenient to change variables from (x, p_x, t) to the new variables (x', H_\perp, τ) defined by [1]

$$x' = x, \quad \tau = t, \quad H_\perp = \frac{1}{2\gamma_b m_b} p_x^2 + \psi^0(x). \quad (18)$$

Substituting Eqs. (18) into Eqs. (16) and (17) gives for the evolution of the perturbations $\delta f_b(x', H_\perp, \tau)$ and $\delta \psi(x', \tau)$,

$$\left(\frac{\partial}{\partial \tau} + v_x \frac{\partial}{\partial x'} \right) \delta f_b = v_x \frac{\partial \delta \psi}{\partial x'} \frac{\partial F_b}{\partial H_\perp}, \quad (19)$$

$$\frac{\partial^2}{\partial x'^2} \delta \psi = -\frac{4\pi e_b^2}{\gamma_b^2} \delta n_b. \quad (20)$$

In Eq. (19), $v_x = +v(H_\perp, x')$ for the forward-moving particles with $v_x > 0$, and $v_x = -v(H_\perp, x')$ for the backward-moving particles with $v_x < 0$, where

$$v_x = \pm v(H_\perp, x') \equiv \pm \left(\frac{2H_\perp}{\gamma_b m_b} \right)^{1/2} \left[1 - \frac{\psi^0(x')}{H_\perp} \right]^{1/2}. \quad (21)$$

Furthermore,

$$\frac{\partial F_b}{\partial H_\perp} = -\frac{\widehat{n}_b}{2\gamma_b m_b \widehat{v}_0} \delta(H_\perp - \widehat{H}_\perp), \quad (22)$$

where $\widehat{v}_0 = (2\widehat{H}_\perp/\gamma_b m_b)^{1/2}$. Using Eqs. (19)-(22) and introducing $\delta E_x(x', \tau) = -(\partial/\partial x')\delta\phi(x', \tau) = -(\gamma_b^2/e_b)(\partial/\partial x')\delta\psi(x', \tau)$, after some algebraic manipulation we obtain [1]

$$\begin{aligned} \frac{\partial^2}{\partial \tau^2} \delta E_x - \widehat{v}_0^2 N(x') \frac{\partial}{\partial x'} \left[N(x') \frac{\partial}{\partial x'} \delta E_x \right] \\ = -\frac{\widehat{\omega}_{pb}^2}{\gamma_b^2} N(x') \delta E_x, \end{aligned} \quad (23)$$

where $N(x')$ is the (dimensionless) profile shape function defined by

$$N(x') = \left[1 - \frac{\psi^0(x')}{\widehat{H}_\perp} \right]^{1/2}. \quad (24)$$

In the analysis of Eq. (23), we make use of a normal-mode approach and express $\delta E_x(x', \tau) =$

$\delta \widehat{E}_x(x', \omega) \exp(-i\omega\tau)$, where ω is the (generally complex) oscillation frequency. Equation (23) can be represented in a convenient form by introducing the angle variable α defined by

$$\alpha = \frac{\pi}{2} \frac{X'}{X_b} = \frac{\omega_0}{\widehat{v}_0} X', \quad (25)$$

where X' and ω_0 are defined by

$$X' = \int_0^{x'} \frac{dx'}{N(x')}, \quad \omega_0 = \frac{\pi}{2} \frac{\widehat{v}_0}{X_b}, \quad (26)$$

where $X_b = X'(x_b)$. Substituting Eq. (25) into Eq. (23) gives the eigenvalue equation

$$\omega_0^2 \frac{\partial^2}{\partial \alpha^2} \delta \widehat{E}_x + \left[\omega^2 - \frac{\widehat{\omega}_{pb}^2}{\gamma_b^2} N(\alpha) \right] \delta \widehat{E}_x = 0. \quad (27)$$

Equation (27) is to be solved over the interval $-\pi/2 < \alpha < \pi/2$ subject to the boundary conditions $\delta \widehat{E}_x(\alpha = \pm\pi/2, \omega) = 0$. Substituting Eqs. (10) and (24) into Eq. (25) gives

$$\alpha = \frac{\pi}{2} \frac{\lambda_D}{X_b} 3^{1/2} \int_N^1 \frac{dz}{[(1-z)(a^+ - z)(z - a^-)]^{1/2}}, \quad (28)$$

where a^\pm is defined in Eq. (12). Some algebraical manipulation gives exactly for the inverse function $N(\alpha)$

$$N(\alpha) = \frac{\left[1 - a^+ \kappa^2 sn^2 \left(\frac{\alpha}{\pi} \frac{X_b}{\lambda_D} \left[\frac{a^+ - a^-}{3} \right]^{1/2}, \kappa \right) \right]}{\left[1 - \kappa^2 sn^2 \left(\frac{\alpha}{\pi} \frac{X_b}{\lambda_D} \left[\frac{a^+ - a^-}{3} \right]^{1/2}, \kappa \right) \right]}, \quad (29)$$

where $sn(\beta, \kappa)$ is the Jacobi elliptic sine function and $\kappa = [(1 - a^+)/(a^+ - a^-)]^{1/2}$. In Eqs. (28)-(29), the "stretched" half-layer thickness (X_b) measured in units of the Debye length (λ_D) is given by

$$\frac{X_b}{\lambda_D} = \frac{2 \cdot 3^{1/2}}{(a^+ - a^-)^{1/2}} F \left(\arcsin(\kappa^2/a^+)^{-1/2}, \kappa \right), \quad (30)$$

where F is the elliptic integral of the first kind. Using the expression for $N(\alpha)$ in Eq. (29), the eigenvalue equation (27) can be solved numerically for $\delta \widehat{E}_x(\alpha, \omega)$ and the eigenvalues ω^2 subject to the boundary conditions $\widehat{E}_x(\alpha = \pm\pi/2, \omega) = 0$. An approximate expression for the eigenvalues of the Schroedinger-like equation (27) can be obtained in the WKB approximation. The Born-Zommerfeld formula, when applied to Eq. (27), gives

$$\frac{\widehat{\omega}_{pb}}{\gamma_b \omega_0} \int_{-\pi/2}^{\pi/2} d\alpha \left[\left(\frac{\gamma_b \omega_m}{\widehat{\omega}_{pb}} \right)^2 - N(\alpha) \right]^{1/2} = \pi m, \quad (31)$$

where ω_m is the m th-mode eigenfrequency with m half-wavelength oscillations of $\delta \widehat{E}_x$ over the layer thickness.

Making use of Eq. (28), the result in Eq. (31) can be rewritten as

$$6^{1/2} \int_0^1 \frac{dz (q_m^2 - z)^{1/2}}{[(1-z)(a^+ - z)(z - a^-)]^{1/2}} = \pi m, \quad (32)$$

where q_m and r are defined by $q_m = \omega_m / (\widehat{\omega}_{pb} / \gamma_b)$ and $r = \kappa [(q_m^2 - a^+) / (q_m^2 - 1)]^{1/2}$. Equation (32) has been

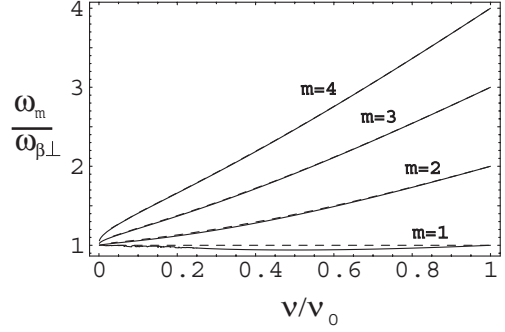


Figure 2: Plots of the normalized mode frequencies $\omega_m/\omega_{\beta\perp}$ versus the on-axis ($x = 0$) tune depression $\nu/\nu_0 = (1 - s_b)^{1/2}$ for several values of mode numbers $m = 1, 2, 3, 4$. The dotted curves are the numerical solutions of the eigenvalue equation (27); the solid curves are the solutions obtained in the WKB approximation [Eq. (32)].

solved numerically [1] for ω_m^2 , and the results have been compared with the numerical solutions of the eigenvalue equation (27) (Fig. 2). In Fig. 2, the convention is such that there are m half-wavelength oscillations of $\delta \widehat{E}_x$ over the layer thickness. Note that low beam intensity ($s_b \ll 1$) corresponds to $\nu/\nu_0 \rightarrow 1$, with $\omega_m \simeq m\omega_{\beta\perp}$, whereas the space-charge-dominated regime ($s_b \rightarrow 1$) corresponds to $\nu/\nu_0 \rightarrow 0$, with $\omega_m \simeq \omega_{\beta\perp} \simeq \widehat{\omega}_{pb}/\gamma_b$.

To summarize, we have demonstrated that the self-consistent waterbag equilibrium f_b^0 satisfying the steady-state ($\partial/\partial t = 0$) Vlasov-Maxwell equations is exactly solvable for the beam density $n_b^0(x)$ and electrostatic potential $\phi^0(x)$. In addition, we derived a closed Schroedinger-like eigenvalue equation for small-amplitude perturbations ($\delta f_b, \delta \phi$) about the self-consistent waterbag equilibrium in Eq. (4). In the eigenvalue equation, the density profile $n_b^0(x)$ plays the role of the potential $V(x)$ in the Schroedinger equation. The eigenvalue equation was investigated analytically and numerically, and the eigenfrequencies were shown to be purely real.

REFERENCES

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- [2] R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001), and references therein.
- [3] The integrals in Eqs. (11), (14), (28), (32) can be expressed in terms of elliptic functions (see Ref. 1).