Abstract

This paper is an analytical investigation of the transverse electron-proton (e-p) two-stream instability in a proton bunch propagating through a stationary electron background. The equations of motion, including the effect of damping, are derived for the centroids of the proton beam and the electron cloud. An attempt is developed to solve the coupled linear centroid equations in the time domain describing the e-p instability in proton bunches with non-uniform line densities. Examples are presented for proton line densities corresponding to uniform and parabolic profiles.

1 INTRODUCTION

For intense proton beams, the focus of recent two-stream instability analyses has been on the transverse instability observed in the Proton Storage Ring (PSR) at Los Alamos National Laboratory [1-4]. The PSR stores a long proton bunch with a near triangular line density profile for a duration of about one millisecond. The instability is observed as rapidly growing transverse oscillations of the stored beam. The instability is observable in the transverse direction for the trapped electrons and the particles are distributed uniformly in the transverse direction so the trapped electrons experience a linear transverse focusing force. The protons are confined in the transverse direction by a linear external focusing force. A Cartesian coordinate system is chosen such that the z axis is pointing opposite to the direction of proton propagation, and the origin coincides with the center of the beam cross section. The line densities of the protons and electrons, \( \lambda_p \) and \( \lambda_e \), generally depend on z.

We consider a bunched proton beam of full length \( L \) and circular cross section of radius \( a \), propagating with a constant velocity \( v \) through a stationary electron background of infinite extent in the direction of beam propagation. We assume that in the equilibrium state, the electrons are trapped in the proton beam, and the particles are distributed uniformly in the transverse direction so the trapped electrons experience a linear transverse focusing force. The protons are confined in the transverse direction by a linear external focusing force. A Cartesian coordinate system is chosen such that the z axis is pointing opposite to the direction of proton propagation, and the origin coincides with the center of the beam cross section. The line densities of the protons and electrons, \( \lambda_p \) and \( \lambda_e \), generally depend on z.

The synchrotron motion of the protons and the axial motion of the electrons in the laboratory frame are neglected for simplicity. Further simplifications are made by neglecting the impedance due to the beam environment, like the beam pipe, and by considering the transverse motion in only one direction, say the \( y \) direction. The stability study is based on a model in which each electron interacts with the proton beam only once, i.e., a “one-pass” interaction between the electrons and protons.

The centroid of the proton beam \( Y_p(z, t) \) and the centroid of electrons \( Y_e(z, t) \) are defined by

\[
Y_q(z, t) = \int_{-\infty}^{\infty} y_q(z, t, \omega_q) F_q(\omega_q)d(\omega_q/\Delta_q),
\]

where the subscripts \( q \) stands for \( p \) (protons) or \( e \) (electrons), \( y_q(z, t, \omega_q) \) is the particle displacement at the position \( z \) and time \( t \), \( \omega_q \) is the oscillation frequency, \( F_q(\omega_q) \) is the frequency distribution function, and \( \Delta_q \) characterizes the frequency spread of \( \omega_q \). The equation of motion for a single particle is

\[
\ddot{y}_q = -\omega_q^2 y_q + (\dot{F}_q/m_q),
\]

where the super-dot denotes derivative with respect to time, \( \dot{F}_q \) is the perturbing force, and \( m_q \) is the relativistic mass of a proton or an electron. For the perturbing forces, we make the approximations \( \dot{F}_q/m_e = \Omega_q^2(z)Y_p \) and \( \dot{F}_q/m_p = G(z)Y_e \), where \( \Omega(z) = (c/a)\sqrt{2r_e\lambda_p(z)} \) is the electron bounce frequency, \( G(z) = \omega_p^2 \xi(z) \), \( \xi(z) = 2r_p c^2 \lambda_e(z)/(\alpha^2 \omega_p^2 \gamma) \), \( c \) is the speed of light, \( \omega_p \) is the betatron frequency, \( \gamma = (1 - v^2/c^2)^{-1/2} \), and \( r_q \) is the classical radius of a proton or an electron.

We consider a Lorentzian distribution function \( F_q(\omega_q) = (\Delta^2_\beta/\pi)[\Delta^2_\beta + (\omega_q - \omega_{q0})^2]^{-1} \), where \( \omega_{q0} \) is the mean value of \( \omega_q \), and assume that the general solution to Eq. (2) can be expressed as a linear combination of \( e^{i\omega_q t} \) and \( e^{-i\omega_q t} \), where \( i = \sqrt{-1} \). Averaging Eq. (2) yields

\[
D^2Y_p + 2\Delta_pDY_p + (\omega_p^2 + \Delta_p^2)Y_p = G(z)Y_e,
\]

where \( D = \omega_p^2 + \Delta_p^2 \).
and

$$\ddot{Y} + 2\Delta \dot{Y} + [\Omega^2(z) + \Delta^2_0]Y = \Omega^2(z)Y_p,$$  \hspace{1cm} (4)

where we have replaced the time derivative in Eq. (2) by \(D = \partial/\partial t - v(\partial/\partial z)\) for the moving proton beam in the laboratory frame. In deriving Eqs. (3) and (4), we have also assumed that the incoherent betatron frequency shift due to the self-fields of the proton beam is negligible, and that the maximum value of \(\lambda_\beta\) is much smaller than that of \(\lambda_\gamma\). Thus, we have \(\omega_{po} = \omega_\beta = \text{depressed betatron frequency}\). Note that the damping factors in Eqs. (3) and (4) depend on the choice of the frequency distribution function.

### 3 STABILITY ANALYSIS

Since the analysis here is based on the linearized equations (3) and (4), the subsequent analysis is necessarily limited to the linear regime. The axial coordinates in the laboratory and beam frames are denoted by \(z\) and \(z'\), respectively. We assume that the origins of these two coordinate systems coincide when \(t = 0\), and that the head of the proton bunch is located at \(z' = 0\). For simplicity, relativistic effects are neglected and the Galilean transformation \(z' = z + vt\) is used between coordinate systems.

Substituting \(Y_p(z', t) = e^{-\Delta_0 t}Y_{en}(z', t)\) and \(Y_{en}(z, t) = e^{-\Delta t}Y_{en}(z, t)\) into Eqs. (3) and (4), and transforming the proton equation to the beam frame, yield

$$\ddot{Y}_{en} + \Omega^2(t - t_e)[Y_{en} - e^{-i(\Delta_0 - \Delta_\beta) t}Y_{en}] = 0,$$  \hspace{1cm} (5)

and

$$\ddot{Y}_{en} + \Omega^2(t - t_e)[Y_{en} - e^{-i(\Delta_0 - \Delta_\beta) t}Y_{en}] = 0,$$  \hspace{1cm} (6)

where a sharp-edged line density \(\lambda_p\) is assumed so that \(\Omega(x) = 0\) for \(x \leq 0\), and \(t_e = -z/v\) is the time when a slice of electrons located at the position \(z\) in the laboratory frame enters the proton bunch. Using the variational method and applying a Fourier transformation, we derive from Eqs. (5) and (6) the integral equation

$$\hat{Y}(z', k) = \int_0^{z'/v} \frac{G(vx)}{\omega_\beta^2 - k^2}W(x) \Phi(x) \Psi(x) \left[ 1 - \frac{2\Omega(x)}{\omega_\beta^2} \right] dx,$$  \hspace{1cm} (7)

where \(\hat{Y}(z', k) = e^{ik(z' - \Delta_\gamma z')}Y(z', k)/G(z'), s' = z'/v \leq L/v\), and \(Y(z', k)\) is the Fourier transform of \(Y_{en}(z, t)\). Here, \(\Phi(t) = \Phi(z, t)\) and \(\Psi(t) = \Psi(z, t)\) are the linearly independent solutions of the homogeneous part of Eq. (6), and \(W(x)\) is the Wronskian of \(\Phi(x)\) and \(\Psi(x)\). In obtaining Eq. (7), we have assumed that \(Y_{en} = dY_{en}/dt = 0\) for \(t \leq t_e\). Differentiating Eq. (7) twice leads to the following equation for \(\hat{Y}(z', k)\), i.e.,

$$v^2 \frac{d^2\hat{Y}}{dz'^2} = -\Omega^2(z') \left[ 1 + \frac{G(z')}{(k^2 - \omega_\beta^2)} \right] \hat{Y}.$$  \hspace{1cm} (8)

Thus, the stability analysis has now resulted in solving Eqs. (7) or (8) and inverting a Fourier transformation. For non-uniform line densities, Eqs. (7) or (8) have exact solutions only for very limited cases. A possible approximation in Eq. (7) can be seen by substituting \(\hat{Y}(z', k) = \zeta(z', k)\Phi(s')\) into Eq. (7), which gives

$$\zeta(z', k) = \int_0^{z'/v} \frac{\Omega^2(x)G(vx)\Phi(x)\Psi(x)}{(\omega_\beta^2 - k^2)W(x)} \times \left[ 1 - \frac{\Phi(s')/\Phi(s')\Psi(x)/\Psi(x)}{dx} \right] dx.$$  \hspace{1cm} (9)

We assume that (i) \(\Phi\) and \(\Psi\) are such that \(\Phi\Psi\) is a smooth function and \(\Phi/\Psi\) is an oscillatory function, and (ii) that the main motion described in \(\zeta(z', k)\) is betatron oscillations at much lower frequency than the electron bounce motion in the beam, so that the contribution from the fast oscillatory term containing \(\Phi/\Psi\) in the integration in Eq. (9) is negligibly small. Then, neglecting the term proportional to \(\Phi/\Psi\) in Eq. (9) leads to the approximate solution

$$Y_p(z', t) \sim \left\{ J(z')/\omega_\beta(t - s') \right\}^{1/4}G(z')\Phi(s') \times \exp \left[ -i(\Delta_\gamma + \Delta_\rho)(t - s') \right] \times \exp \left[ -i(4/3)J(z')(t - s') \right],$$  \hspace{1cm} (10)

where

$$J(z') = i \int_0^{z'/v} \frac{\Omega^2(x)G(vx)}{\omega_\beta^2 W(x)} \Phi(x)\Psi(x) dx.$$  \hspace{1cm} (11)

Similarly, the solution for \(Y_{en}(z', t)\) in terms of \(\Psi(s')\) can be derived by making the substitution \(Y(z', k) = \zeta(z', k)\Phi(s')\) in Eq. (7). Equation (10) indicates that the instability grows in both time and space consistent with computer simulation results[11]. In the absence of damping, the asymptotic temporal growth of the instability in the beam frame is proportional to \(e^{\omega_{s} t^2/2}\) (where the constant \(\omega_s\) is independent of \(t\)), and the spatial growth is determined primarily by the quantity \(J(z')\). Further, we see from Eq. (10) that the \(e-p\) mode “wiggles” in space proportional to \(\kappa e^{-iJ/4}\). Note that Eq. (10) also indicates that the perturbation is eventually damped as long as \(\Delta_\gamma\) is nonzero. This is due to the combined effects of finite proton bunch length and the “one-pass” electron-proton interaction.

The following two examples illustrate the applications of the theory developed here.

**Example A: Uniform Line Densities \(\lambda_p\) and \(\lambda_\epsilon\)**

When both \(\lambda_p\) and \(\lambda_\epsilon\) are uniform, \(\Omega^2(z) = \omega^2\), and \(G(x) = \xi \omega_\beta^2\), where \(\omega\) is the electron bounce frequency, and \(\xi\) is a constant. Equations (7) and (8) have the exact solutions \(\Phi(z') = \exp(\xi s\omega s'\eta)\) and \(\Psi(z') = \exp(-\xi s\omega s'\eta)\), where \(\eta = \left[ 1 + \left( \xi \omega_\beta^2/\left(k^2 - \omega_\beta^2\right) \right) \right]^{1/2}\). We concentrate on the part of the solution containing \(\Phi\) here. The part containing \(\Psi\) can be treated in the same manner. The solution for \(Y_p(z', t)\) is

$$Y_p(z', t) \sim \xi \omega_\beta^2 e^{\Delta_\gamma(t - s')} \int_{-\infty}^{\infty} e^{i \left( k(t - s') + \omega s' \eta \right) d\eta},$$  \hspace{1cm} (12)

where \(z' \leq L\). The inverse Fourier transformation in Eq. (12) can be carried out analytically only in a few parameter ranges by using the steepest descent method. We limit discussion here to two cases.

The first case corresponds to \(\xi^2 \ll \xi s\omega s'/\left(\omega_\beta(t - s')\right) \ll 1\). In this range, the unstable mode has the approximate solution...
where we have neglected constant factors in the amplitude. This solution agrees well with the result of applying the approximation in Eq. (10). Equation (13) indicates that the proton beam “wiggles” in space approximately according to the electron bounce frequency as seen in experiments.[1-4]. It can be shown that if the damping is large enough, the instability grows to a maximum amplitude and then subsequently damps.

The second case corresponds to $\xi^2 s^2/|\omega_\beta(t-s')|^2 \ll \xi^3 \ll 1$. The approximate solution for the unstable mode in this range is found to be

$$Y_p(z', t) \sim \{\omega s'/[\omega_\beta(t-s')]^3\}^{1/4} \exp[-\Delta_e s' - i\omega_\beta t + i(\omega + \omega_\beta)s' - \Delta_p(t-s') + \sqrt{\xi \omega_\beta s'(t-s')}],$$

where only two lowest-order terms in the expansion of $R^2$ have been retained. The solution for $Y_p(z', t)$ can be obtained by substituting Eqs. (16) and (19) into Eq. (10). For no damping, the combined spatial and temporal growths at the center and the tail of the bunch are approximately $\exp[\xi \omega_\beta t^{1/2} h^{3/4}]$ and $\exp[2.3(\xi \omega_\beta t^{1/2} h^{3/4})]$, respectively.

### 4 CONCLUSIONS

We have derived the equations for the transverse motion of the centroids of the proton bunch and the electrons. The damping effect was included by considering Lorentzian distributions of the particles’ transverse oscillation frequencies. Based on the model of “one-pass” interaction between the protons and the electrons, we discussed approaches for solving the linearized centroid equations to investigate the $e-p$ instability in finite-length proton bunches with arbitrary line density. Case studies were presented for uniform and parabolic proton line densities. It was found that the $e-p$ instability can grow in both space and time. Depending on the stage of the instability, the temporal growth of the electron-proton oscillations can be proportional to $e^{\omega t^{1/2}}$ or $e^{\omega t^{1/3}}$. For a Lorentzian distribution of proton oscillation frequencies with nonzero spread, the instability will eventually be damped for long times.

### 5 REFERENCES