RENORMALIZATION GROUP APPROACH TO THE BEAM-BEAM INTERACTION IN CIRCULAR COLLIDERS

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Abstract

Building on the Renormalization Group (RG) method the beam-beam interaction in circular colliders is studied. A regularized symplectic RG beam-beam map, that describes successfully the long-time asymptotic behavior of the original system has been obtained. The integral of motion possessed by the regularized RG map has been used to construct the invariant phase space density (stationary distribution function), and a coupled set of nonlinear integral equations for the distributions of the two colliding beams has been derived.

1 INTRODUCTION

The problem of coherent beam-beam interaction in storage ring colliders is one of the most important, and at the same time one of the most difficult problems in contemporary accelerator physics. Its importance lies in the fact that beam-beam interaction is the basic factor, limiting the luminosity of a circular collider. Nevertheless, some progress in the analytical treatment of the coherent beam-beam interaction has been made [1] - [3], it is still far from being completely understood. In most of the references available the basic trend of analysis follows the perturbative solution of the Vlasov-Poisson equations, where the linearized system is cast in the form of an eigenvalue problem for the eigenmodes.

An important question, which still remains unanswered is how to determine the invariant phase space density (equilibrium distribution function) if such exist. In the present paper we develop a novel approach to the beam-beam interaction in circular colliders, based on the Renormalization Group (RG) method [4]. The basic idea of the RG method is to remove secular or divergent terms by renormalizing the integration constants of the lowest order perturbative solution. Its extension to discrete symplectic maps is however not straightforward, and should be performed with care. Here we follow the regularization procedure outlined in the paper by Goto and Nozaki [5].

2 THE NONLINEAR BEAM-BEAM MAP

We begin with the one-dimensional model of coherent beam-beam interaction in the vertical (q) direction described by the Hamiltonian

$$H_k = \frac{\dot{\chi}_k}{2}\left(p^2 + q^2\right) + \lambda_k \delta_p(\theta) V_k(q;\theta), \tag{2.1}$$

where the normalized beam-beam potential $V_k(q;\theta)$ satisfies the Poisson equation

$$\frac{\partial^2 V_k}{\partial q^2} = 4\pi \int\limits_{-\infty}^{\infty} dp f_{3-k}(q,p;\theta), \tag{2.2}$$

and

$$\lambda_k \simeq \frac{2 R r_e N_{3-k} \beta_{kq}^*}{\gamma_{k0} L_{(3-k)x}}. \tag{2.3}$$

Here, $(k = 1, 2)$ labels the counter-propagating beams, θ is the azimuthal angle, $\dot{\chi}_k = R\beta_{kq}^{-1}$ is the derivative of the phase advance with respect to θ, R is the mean machine radius, r_e is the classical electron radius, $N_{1,2}$ is the total number of particles in either beam, β_{kq}^* is the vertical beta-function at the interaction point, and L_{kx} is the horizontal dimension of the beam ribbon. In addition, the distribution function $f_k(q,p;\theta)$ is a solution to the Vlasov equation

$$\frac{\partial f_k}{\partial \theta} + \dot{\chi}_k p \frac{\partial f_k}{\partial q} - \frac{\partial H_k}{\partial q}\frac{\partial f_k}{\partial p} = 0. \tag{2.4}$$

In order to build the iterative beam-beam map, we formally solve the Hamilton's equations of motion. As a result, we obtain

$$\begin{aligned} q_{n+1} &= q_n \cos\omega_k + \left[p_n - \lambda_k V_k'(q_n)\right]\sin\omega_k, \\ p_{n+1} &= -q_n \sin\omega_k + \left[p_n - \lambda_k V_k'(q_n)\right]\cos\omega_k, \end{aligned} \tag{2.5}$$

where the prime implies differentiation with respect to the spatial variable q, and $\omega_k = 2\pi\nu_k$.

3 RENORMALIZATION GROUP REDUCTION OF THE BEAM-BEAM MAP

Multiplying the first of Eqs. (2.5) by $\cos\omega_k$, multiplying the second one by $-\sin\omega_k$, and summing up the two equations, we find

$$q_{n+1}\cos\omega_k - p_{n+1}\sin\omega_k = q_n. \tag{3.1}$$

Using Eq. (3.1) a second order difference equation

$$\widehat{\mathcal{L}} q_n = -\epsilon \lambda_k V_k'(q_n)\sin\omega_k \tag{3.2}$$

can be obtained. Here $\widehat{\mathcal{L}} q_n = q_{n+1} - 2q_n\cos\omega_k + q_{n-1}$, and ϵ is a formal small parameter (set to unity at the end of the calculations), taking into account the fact that the beam-beam kick is small and can be treated as perturbation.

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Next we consider an asymptotic solution of the map (3.2) for small ϵ by means of the RG method. The naive perturbation expansion

$$q_n = q_n^{(0)} + \epsilon q_n^{(1)} + \epsilon^2 q_n^{(2)} + \cdots \tag{3.3}$$

when substituted into Eq. (3.2) yields the perturbation equations order by order

$$\widehat{\mathcal{L}} q_n^{(0)} = 0, \qquad \widehat{\mathcal{L}} q_n^{(1)} = -\lambda_k V_k' \left(q_n^{(0)} \right) \sin\omega_k, \tag{3.4}$$

$$\widehat{\mathcal{L}} q_n^{(2)} = -\lambda_k q_n^{(1)} V_k'' \left(q_n^{(0)} \right) \sin\omega_k, \tag{3.5}$$

Solving the first of Eqs. (3.4) for the zeroth order contribution, we obtain the obvious result

$$q_n^{(0)} = A_k e^{i\omega_k n} + \text{c.c.} = 2|A_k| \cos(\omega_k n + \phi_k), \tag{3.6}$$

where A_k is a complex integration constant, whose amplitude and phase are $|A_k|$ and ϕ_k, respectively. Let us assume for the time being that the beam-beam potential $V_k(q)$ is a known function of the vertical displacement q. Then, we have

$$V_k' \left(q_n^{(0)} \right) = \sum_{M=1}^{\infty} \mathcal{C}_k^{(M)} A_k^{2M-1} e^{i(2M-1)\omega_k n} + \text{c.c.} \tag{3.7}$$

Here the coefficients $\mathcal{C}_k^{(M)}$ are functions of the amplitude $|A_k|$, given by the expression

$$\mathcal{C}_k^{(M)}(|A_k|) = \frac{1}{\pi} \frac{(-1)^M}{|A_k|^{2M-1}} \int_0^{\infty} d\lambda \lambda V_k(\lambda) \times \mathcal{J}_{2M-1}(2\lambda |A_k|), \tag{3.8}$$

where $V_k(\lambda)$ is the Fourier image of the beam-beam potential and $\mathcal{J}_m(z)$ is the Bessel function of the first kind of order m. Similarly for the second derivative of the beam-beam potential $V_k''\left(q_n^{(0)} \right)$, entering the second order perturbation equation (3.5), we obtain

$$V_k'' \left(q_n^{(0)} \right) = \mathcal{D}_k^{(0)} + \sum_{M=1}^{\infty} \mathcal{D}_k^{(M)} A_k^{2M} e^{i2M\omega_k n} + \text{c.c.}, \tag{3.9}$$

where

$$\mathcal{D}_k^{(0)}(|A_k|) = -\frac{1}{\pi} \int_0^{\infty} d\lambda \lambda^2 V_k(\lambda) \mathcal{J}_0(2\lambda |A_k|), \tag{3.10}$$

$$\mathcal{D}_k^{(M)}(|A_k|) = \frac{1}{\pi} \frac{(-1)^{M+1}}{|A_k|^{2M}} \int_0^{\infty} d\lambda \lambda^2 V_k(\lambda) \times \mathcal{J}_{2M}(2\lambda |A_k|). \tag{3.11}$$

From the recursion property of Bessel functions [6, 7]

$$\mathcal{J}_{\nu-1}(z) + \mathcal{J}_{\nu+1}(z) = \frac{2\nu}{z} \mathcal{J}_\nu(z) \tag{3.12}$$

we deduce an important relation to be used later

$$\mathcal{D}_k^{(N)} - \mathcal{D}_k^{(N+1)} |A_k|^2 = (2N+1) \mathcal{C}_k^{(N+1)}. \tag{3.13}$$

The solutions of the perturbation equations (3.4) and (3.5), taking into account (3.13) are given by

$$q_n^{(1)} = \frac{i\lambda_k n}{2} \mathcal{C}_k^{(1)} A_k e^{i\omega_k n} + \frac{\lambda_k \sin\omega_k}{2} \times \sum_{M=1}^{\infty} \widetilde{\mathcal{C}}_k^{(M+1)} A_k^{2M+1} e^{i(2M+1)\omega_k n} + \text{c.c.}, \tag{3.14}$$

where for brevity the explicit form of $q_n^{(2)}$ is not reproduced, and

$$\widetilde{\mathcal{C}}_k^{(N+1)} = \frac{\mathcal{C}_k^{(N+1)}}{\cos\omega_k - \cos(2N+1)\omega_k}. \tag{3.15}$$

To remove secular terms, proportional to n and n^2 (in $q_n^{(1)}$ and $q_n^{(2)}$), we define the renormalization transformation $A_k \rightarrow \widetilde{A}_k(n)$ by collecting all terms proportional to the fundamental harmonic $e^{i\omega_k n}$. Solving perturbatively the resulting equation, we can express A_k in terms of $\widetilde{A}_k(n)$. A discrete version of the RG equation can be defined by considering the difference $\widetilde{A}_k(n+1) - \widetilde{A}_k(n)$. Substituting the expression for A_k in terms of $\widetilde{A}_k(n)$ into the above mentioned difference, we can eliminate the secular terms up to $O(\epsilon^2)$. The result is

$$\widetilde{A}_k(n+1) = \left[1 + \epsilon \frac{i\lambda_k}{2} \mathcal{C}_k^{(1)} - \epsilon^2 \frac{\lambda_k^2}{8} \mathcal{C}_k^{(1)\mathbf{2}} (1 + i\cot\omega_k) + i\epsilon^2 \frac{\lambda_k^2 \sin\omega_k}{4} \sum_{N=1}^{\infty} \widetilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} \left| \widetilde{A}_k(n) \right|^{4N} \right] \widetilde{A}_k(n). \tag{3.16}$$

This naive RG map does not preserve the symplectic symmetry and does not have a *constant of motion*. To recover the symplectic symmetry we regularize the naive RG map by noting that the coefficient in the square brackets, multiplying $\widetilde{A}_k(n)$ can be exponentiated:

$$\widetilde{A}_k(n+1) = \widetilde{A}_k(n) \exp\left[i\widetilde{\omega}_k \left(\left| \widetilde{A}_k(n) \right| \right) \right], \tag{3.17}$$

where

$$\widetilde{\omega}_k \left(\left| \widetilde{A}_k(n) \right| \right) = \epsilon \frac{\lambda_k \mathcal{C}_k^{(1)}}{2} + \epsilon^2 \frac{\lambda_k^2}{8} \left(-\mathcal{C}_k^{(1)\mathbf{2}} \cot\omega_k + 2\sin\omega_k \sum_{N=1}^{\infty} \widetilde{\mathcal{C}}_k^{(N+1)} \mathcal{D}_k^{(N)} \left| \widetilde{A}_k(n) \right|^{4N} \right). \tag{3.18}$$

It is clear now that the regularized RG map (3.17) possesses the obvious integral of motion:

$$\left| \widetilde{A}_k(n+1) \right| = \left| \widetilde{A}_k(n) \right| = \sqrt{\frac{J_k}{2}}. \tag{3.19}$$

Proceeding in the same way as above and making use of the relation (3.1), we can obtain the renormalized map for the canonical conjugate momentum p_n. As a result the integral of motion J_k can be represented in the form of a generalized Courant-Snyder invariant, and can be written as

$$2J_k = q^2 + \frac{[p - \alpha_k(J_k)q]^2}{\beta_k^2(J_k)}. \tag{3.20}$$

It is important to emphasize that Eq. (3.20) comprises a transcendental equation for the invariant J_k as a function of the canonical variables (q, p), since the coefficients α_k and β_k depend on J_k themselves.

4 THE INVARIANT PHASE SPACE DENSITY

If an integral of motion J_k of the beam-beam map (2.5) exists, it can be proved that the invariant phase space density $f_k^{(I)}(q, p)$ [which is a solution to the Vlasov equation (2.4)] is a generic function of J_k, that is

$$f_k^{(I)}(q, p) = F_k(J_k) \qquad (k = 1, 2). \tag{4.1}$$

Here $F_k(z)$ is a generic function of its argument. Since the integral of motion J_k is a functional of the invariant density of the opposing beam $f_{3-k}^{(I)}(q, p)$, Eq. (4.1) comprises a coupled system of nonlinear integral equations for the invariant densities of the two counter-propagating beams. Let us find the integral of motion [see Eq. (3.20)] up to first order in the perturbation parameter ϵ. We have

$$J_k = J_0 - \frac{\lambda_k \mathcal{C}_k^{(1)}(J_0)}{2}\left(p^2 \cot\omega_k + pq\right), \tag{4.2}$$

where

$$J_0 = \frac{1}{2}\left(p^2 + q^2\right). \tag{4.3}$$

The Fourier image of the beam-beam potential

$$V_k(\lambda) = -\frac{4\pi}{\lambda^2} \int\limits_{-\infty}^{\infty} dq' \int\limits_{-\infty}^{\infty} dp' f_{3-k}^{(I)}(q', p') \cos\lambda q', \tag{4.4}$$

obtained by solving the Poisson equation (2.2) is next substituted into the corresponding expression [see Eq. (3.8)] for the coefficient $\mathcal{C}_k^{(1)}(J_0)$. Taking into account the recursion relation (3.12) as well as the identity [6]

$$\int\limits_0^{\infty} dx \mathcal{J}_{2n}(x) \cos ax = \frac{(-1)^n T_{2n}(a)}{\sqrt{1 - a^2}} \qquad [0 < a < 1], \tag{4.5}$$

where $T_n(z)$ is the Chebyshev polynomial of order n, we obtain

$$\mathcal{C}_k^{(1)}(J_0) = \frac{8}{J_0} \int\limits_{-\infty}^{\infty} dp' \int\limits_0^{\sqrt{2J_0}} dq' f_{3-k}^{(I)}(q', p') \sqrt{2J_0 - q'^2}. \tag{4.6}$$

Thus, we finally arrive at the system of integral equations for the invariant phase space densities $f_k^{(I)}(q, p)$

$$f_k^{(I)}(q, p) = C_k F_k \left[J_0 - \frac{4\lambda_k}{J_0}\left(p^2 \cot\omega_k + pq\right) \times \int\limits_{-\infty}^{\infty} dp' \int\limits_0^{\sqrt{2J_0}} dq' f_{3-k}^{(I)}(q', p') \sqrt{2J_0 - q'^2} \right], \tag{4.7}$$

where

$$C_k = \left[\int\limits_{-\infty}^{\infty} dp \int\limits_{-\infty}^{\infty} dq F_k(q, p) \right]^{-1}. \tag{4.8}$$

5 CONCLUDING REMARKS

As a result of the investigation performed, we have obtained a regularized symplectic RG beam-beam map, that describes correctly the long-time asymptotic behavior of the original system. It has been shown that the regularized RG map possesses an integral of motion, which can be determined to any desired order. The invariant phase space density (stationary distribution function) has been constructed as a generic function of the integral of motion, and a coupled set of nonlinear integral equations for the distributions of the two colliding beams has been derived.

It is worthwhile to note that the method presented here is also applicable to study the four-dimensional symplectic beam-beam map, governing the dynamics of counter-propagating beams in the plane transverse to the particle orbit.

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