Symmetries and Invariants of the Time-Dependent Oscillator Equation and the Envelope Equation*

Hong Qin and Ronald C. Davidson
Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543, USA

Abstract

The single-particle dynamics in a time-dependent focusing field is examined. The existence of the Courant-Snyder invariant is fundamentally a result of the corresponding symmetry admitted by the oscillator equation with time-dependent frequency. A careful analysis of the admitted symmetries reveals a deeper connection between the nonlinear envelope equation and the oscillator equation. A general theorem regarding the symmetries and invariants of the envelope equation, which includes the existence of the Courant-Snyder invariant as a special case, is demonstrated. The symmetries of the envelope equation enable a fast algorithm for finding matched solutions without using the conventional iterative shooting method.

INTRODUCTION

The Courant-Snyder invariant for an oscillator with time-dependent frequency is an important concept for accelerator physics [1]. For an oscillation amplitude $u(t)$ satisfying

$$\ddot{u} + \kappa(t)u = 0\,, \tag{1}$$

where $\kappa(t)$ is the time-dependent frequency coefficient, the Courant-Snyder invariant is given by [2]

$$I = \frac{u^2}{w^2} + (\dot{w}u - w\dot{u})^2\ . \tag{2}$$

Here, $w = w(t)$ is any solution of the envelope equation

$$\ddot{w} + \kappa(t)w - \frac{1}{w^3} = 0\,. \tag{3}$$

This classical result has been derived many times using different methods. Initially, it was derived by Courant and Snyder in 1958 [2] using the basic techniques for Hill's equation. It was rediscovered by Lewis [3] using the asymptotic method developed by Kruskal [4]; Eliezer and Gray [5] demonstrated a physical interpretation of the invariant; a derivation using linear canonical transformation was given by Leach [6]; and Lutzky re-derived the result using Noether's theorem [7]. A short review of various derivation methods can be found in Ref. [8]. We note that the basic concept of the Courant-Snyder invariant may had appeared earlier in other formats. For example, Kulsrud obtained two equations for w which are equivalent to Eq. (3) [9]. The concept of an envelope function w and its notation, we believe, can be attributed to a paper by Brikhoff [10], which predated the 1911 Solvay Conference, where, according to commonly accepted history, the concept of adiabatic invariant for a time-dependent harmonic oscillator was first discussed by Lorentz and Einstein [11].

In this paper, we first re-examine the time-dependent harmonic oscillator equation from the viewpoint of the symmetry group G for Eq. (1). It is shown that the symmetry group for Eq. (1) is generated by an 8D Lie algebra (infinitesimal generator) g, which contains the 3D subalgebra g_{CS} that corresponds to the Courant-Snyder invariant. The envelope equation appears naturally as the determining equation for g_{CS}. We then investigate the symmetry group of the envelope equation itself. It is interesting that the determining equation for the Lie algebra g_w of the symmetry group G_w for the envelope equation is an envelope equation itself. A theorem regarding the symmetry and the invariant for envelope equations is presented, together with applications.

SYMMETRY GROUP FOR TIME-DEPENDENT OSCILLATOR EQUATION

A symmetry group can be used to reduce the order of differential equations and to generate invariants [12]. We search for vector fields v in (t, u) space

$$v = \xi(t,u)\frac{\partial}{\partial t} + \phi(t,u)\frac{\partial}{\partial u} \tag{4}$$

as infinitesimal generators (Lie algebra) g for the symmetry transformation group G, which leaves Eq. (1) invariant. The vector field v will induce a vector field in $(t, u, \dot{u}, \ddot{u})$ space, i.e., the prolongation of v denoted by $pr^{(2)}v$,

$$pr^{(2)}v = \xi\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^u\frac{\partial}{\partial \dot{u}} + \phi^{uu}\frac{\partial}{\partial \ddot{u}}\,, \tag{5}$$

$$\phi^u \equiv \phi_t + (\phi_u - \xi_t)\dot{u} - \xi_u\dot{u}^2\,, \tag{6}$$

$$\phi^{uu} \equiv -3\xi_u\dot{u}\ddot{u} + (\phi_u - 2\xi_t)\ddot{u} - \xi_{uu}\dot{u}^3 + (\phi_{uu} - 2\xi_{tu})\dot{u}^2 + (2\phi_{ut} - \xi_{tt})\dot{u} + \phi_{tt}\ . \tag{7}$$

The determining equation for v to be an infinitesimal generator for G is

$$pr^{(2)}v\,[\ddot{u} + \kappa(t)u] = \phi^{uu} + \kappa\phi + \xi\dot{\kappa}u = 0\,. \tag{8}$$

Substituting the expression for ϕ^{uu}, we obtain

$$-\xi_{uu}\dot{u}^3 + (\phi_{uu} - 2\xi_{tu})\dot{u}^2 + (3\kappa\xi_u u + 2\phi_{ut} - \xi_{tt})\dot{u} - (\phi_u - 2\xi_t)\kappa u + \phi_{tt} + \kappa\phi + \dot{\kappa}\xi u = 0\,. \tag{9}$$

* Research supported by the U.S. Department of Energy. We thank Drs. K. Takayama, T.-S Wang, P. Channell, and R. Kulsrud for many productive discussions.

Since Eq. (9) should be valid everywhere in $(t, u, \dot{u})$ space, the coefficients of $\dot{u}^3$, $\dot{u}^2$, and $\dot{u}$ should vanish, i.e.,

$$\xi_{uu} = 0\,, \tag{10}$$

$$\phi_{uu} - 2\xi_{tu} = 0\,, \tag{11}$$

$$3\kappa\xi_u u + 2\phi_{ut} - \xi_{tt} = 0\,, \tag{12}$$

$$-\kappa\left(\phi_u - 2\xi_t\right)u + \phi_{tt} + \kappa\phi + \dot{\kappa}\xi u = 0\,. \tag{13}$$

Equations (10)-(13) can be used to find the solutions for ξ and ϕ. After some algebra, we obtain

$$\xi = a(t)u + b(t)\,, \tag{14}$$

$$\phi = \dot{a}(t)u^2 + c(t)u + d(t)\,, \tag{15}$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ satisfy

$$\ddot{a} + \kappa a = 0\,, \tag{16}$$

$$\ddot{d} + \kappa d = 0\,, \tag{17}$$

$$\dddot{b} + 4\kappa\dot{b} + 2\dot{\kappa}b = 0\,, \tag{18}$$

$$\dot{c} - \frac{\ddot{b}}{2} = 0\,. \tag{19}$$

Equations (16)-(19) have eight degrees of freedom. Therefore, the Lie algebra g is 8D, which is the maximum dimension that a second-order ODE can have for the Lie algebra of its symmetry group. The sub-algebras generated by a, d, and b are independent, and have the dimension of 2, 2, and 3 respectively. From Eq. (19), we obtain

$$c = \frac{\dot{b}}{2} + c_0\,. \tag{20}$$

There is one degree of freedom associated with c_0.

According to the basic result of Noether's theorem, every infinitesimal divergence symmetry corresponds to an invariant [12]. Here, an infinitesimal divergence symmetry is defined as a vector field satisfying

$$pr^{(2)}v(L) + L\frac{d\xi}{ds} = \frac{dB(t,u)}{dt} \tag{21}$$

for some function $B(t,u)$. In Eq. (21), L is the Lagrangian for Eq. (1). It can be shown that

$$pr^{(2)}v(L) = \frac{dA}{dt} + \xi\frac{dL}{dt} \tag{22}$$

for some function $A(t,u)$, from which it follows that $I = B - A - L\xi$ is an invariant if v is an infinitesimal divergence symmetry. It can also be demonstrated that every infinitesimal divergence symmetry belongs to the Lie algebra g for the symmetry group G of Eq. (1). Since we have obtained the Lie algebra g, to determine all of the invariants of Eq. (1), it is only necessary to verify which subspace of g consists of infinitesimal divergence symmetries. It turns out that the infinitesimal divergence symmetries form a 5D subspace g_1 of the 8D Lie algebra g. It is given by

$$v = b(t)\frac{\partial}{\partial t} + \left[\frac{\dot{b}(t)}{2}u + d(t)\right]\frac{\partial}{\partial u}\,. \tag{23}$$

For the 2D sub-algebra $v = d\partial/\partial u$ associated with d, it is easy to show that the invariant is

$$I = u\dot{d} - \dot{u}d\,, \tag{24}$$

which is the well-known Wronskian for linear equations. For the 3D Lie algebra $v = b\partial/\partial\xi + u(\dot{b}/2)\partial/\partial u$ associated with b, the invariant is found to be

$$I = \left[\frac{\ddot{b}}{4} + \frac{\kappa}{2}b\right]u^2 + \frac{b}{2}\dot{u}^2 - \frac{\dot{b}}{2}u\dot{u}\,. \tag{25}$$

We now show that this is indeed the Courant-Snyder invariant. Let $b = 2w^2$, Eq. (18) becomes

$$w\dddot{w} + 3\dot{w}\ddot{w} + 4\kappa w\dot{w} + \dot{\kappa}w^2 = 0\,, \tag{26}$$

which is equivalent to

$$3\dot{w}h + \dot{h}w = 0\,, \tag{27}$$

$$h \equiv \ddot{w} + \kappa w - \frac{1}{w^3}\,. \tag{28}$$

In other words,

$$h = \frac{\varepsilon - 1}{w^3}$$

for an arbitrary constant ε. Thus, we obtain the envelope equation

$$\ddot{w} + \kappa w - \frac{\varepsilon}{w^3} = 0\,. \tag{29}$$

In terms of w, the infinitesimal generator is

$$v_{CS} = 2w^2\frac{\partial}{\partial t} + 4w\dot{w}u\frac{\partial}{\partial u}\,, \tag{30}$$

and the invariant in Eq. (25) becomes the familiar Courant-Snyder invariant

$$I = (\dot{w}^2 + \frac{\varepsilon}{w^2})u^2 + w^2\dot{u}^2 - 2w\dot{w}u\dot{u}\,. \tag{31}$$

In this sense, we can refer to the symmetry group generated by the infinitesimal generator in Eq. (30) as Courant-Snyder symmetry. The Lie algebra of the Courant-Snyder symmetry is 3D because ε is an arbitrary constant in addition to the two arbitrary constants needed to specify a particular solution for w. Not surprisingly, Eq. (18) is exactly the same as that for the well-known β function in Courant-Snyder theory.

The 3D subspace in g complementary to g_1 does not produce any invariant. The one degree of freedom associated with c_0 in Eq. (20) corresponds to

$$v = c_0 u\frac{\partial}{\partial u}\,,$$

which generates the symmetry group of the scaling transformation $\tilde{u} = \exp(c_0\tau)u$, which is obviously due to the fact that Eq. (1) is linear. The sub-algebra of g generated by a has 2 degrees of freedom, but currently it does not seem to have any appreciable importance.

SYMMETRY GROUP FOR THE ENVELOPE EQUATION

We now apply the symmetry group analysis to the envelope equation [Eq. (29)] itself. The symmetry group G_w for Eq. (29) should be a subgroup of the symmetry group G for Eq. (1), because the special case of Eq. (29) for $\varepsilon = 0$ is Eq. (1). Carrying out a similar procedure to that for deriving Eqs. (16)-(19), we obtain the Lie algebra g_w for G_w as

$$v_w = 2w_1^2 \frac{\partial}{\partial t} + 4w_1 \dot{w}_1 \frac{\partial}{\partial w}\,, \tag{32}$$

where w_1 satisfies another envelope equation

$$\ddot{w}_1 + \kappa w_1 - \frac{\varepsilon_1}{w_1^3} = 0 \tag{33}$$

with an arbitrary constant ε_1. Further analysis shows that v_w is an infinitesimal divergence symmetry with the invariant

$$I = \varepsilon \left(\frac{w_1}{w}\right)^2 + \varepsilon_1 \left(\frac{w}{w_1}\right)^2 + (w\dot{w}_1 - \dot{w}w_1)^2\ . \tag{34}$$

We summarize the above result in the following theorem.

Theorem 1. *For an arbitrary function $\kappa(t)$ and w_1, w_2 satisfying*

$$\ddot{w}_1 + \kappa w_1 = \frac{\varepsilon_1}{w_1^3}\,, \tag{35}$$

$$\ddot{w}_2 + \kappa w_2 = \frac{\varepsilon_2}{w_2^3}\,, \tag{36}$$

where ε_1 and ε_2 are real constants, the quantity

$$I = \varepsilon_1 \left(\frac{w_2}{w_1}\right)^2 + \varepsilon_2 \left(\frac{w_1}{w_2}\right)^2 + (w_2\dot{w}_1 - \dot{w}_2 w_1)^2 \tag{37}$$

is an invariant.

This result was obtained by Lutzky in a less general form [7], and it can be straightfowardly verified by direct calculation. The invariant in Eq. (37) allows us to solve for the general solutions for w_1 in terms of a special solution for w_2. Let $q = w_1/w_2$, we obtain

$$I = \varepsilon_1 \frac{1}{q^2} + \varepsilon_2 q^2 + \left(\frac{dq}{d\psi}\right)^2\,, \tag{38}$$

$$\psi \equiv \int \frac{1}{w_2^2} dt\,. \tag{39}$$

Equation (38) can be solved for q in terms of ψ as

$$q^2 = \frac{I - \sqrt{I^2 - 4\varepsilon_1\varepsilon_2}\sin\left[-2\sqrt{\varepsilon_2}(\psi + C)\right]}{2\varepsilon_2}\,, \tag{40}$$

or equivalently,

$$w_1 = w_2 \left(\frac{I - \sqrt{I^2 - 4\varepsilon_1\varepsilon_2}\sin\left[-2\sqrt{\varepsilon_2}(\psi + C)\right]}{2\varepsilon_2}\right)^{1/2}. \tag{41}$$

Here, I and C are constants. Equation (41) recovers the Courant-Snyder theory, Eqs. (1) and (3), as a special case when $\varepsilon_1 = 0$, and $\varepsilon_2 = 1$. Another application of Theorem 1 and Eq. (41) is in the numerical solution of the envelope equation [Eq. (3)]. For a periodic focusing lattice $\kappa(t)$, it is desirable to find matched solutions to construct the β functions. Normally, this is done by a shooting method, where Eq. (3) is solved numerically many times, iteratively. Using Eq. (41) for the case where $\varepsilon_1 = \varepsilon_2 = 1$, we can have a much more efficient algorithm, where Eq. (3) needs to be numerically solved only once. First, we pick arbitrary initial conditions for $w(t=0) = w_0$ and $\dot{w}(t=0) = \dot{w}_0$ at $t = 0$, and solve numerically for w from $t = 0$ to one lattice period at $t = T$. Denote this solution as $w_s(t)$. Applying Eq. (41), the general solution for w_g is

$$w_g = w_s \left(\frac{I - \sqrt{I^2 - 4}\sin\left[-2(\psi + C)\right]}{2}\right)^{1/2}\,, \tag{42}$$

$$\psi = \int_0^t \frac{1}{w_s^2} dt\,. \tag{43}$$

By selecting I and C such that

$$w_g(0) = w_g(T) \text{ and } \dot{w}_g(0) = \dot{w}_g(T)\,, \tag{44}$$

we obtain the matched solution to Eq. (3) for a periodic focusing lattice $\kappa(t) = \kappa(t + T)$.

REFERENCES

[1] R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators*, World Scientific, Singapore, 2001.

[2] E. Courant and H. Snyder, Annals of Physics **3**, 1 (1958).

[3] J. Lewis, H.R., Journal of Mathematical Physics **9**, 1976 (1968).

[4] M. Kruskal, Journal of Mathematical Physics **3**, 806 (1962).

[5] C. Eliezer and A. Gray, SIAM Journal on Applied Mathematics **30**, 463 (1976).

[6] P. Leach, Journal of Mathematical Physics **18**, 1608 (1977).

[7] M. Lutzky, Physics Letters **68A**, 3 (1978).

[8] K. Takayama, IEEE Transactions on Nuclear Science **NS-30**, 2412 (1983).

[9] R. Kulsrud, Physical Review **106**, 205 (1957).

[10] G. Birkhoff, Trans. American Mathematical Society **9**, 219 (1908).

[11] H. Goldstein, *Classical Mechanics*, pages 531–540, Addison-Wesley, Reading, 1980.

[12] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.