Advances in Plasma Turbulence

after the 1972 publication

of Davidson's book

Methods in Nonlinear Plasma Theory [a]

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Introduction

Ron's 1972 book

Methods in Nonlinear Plasma Theory

was a milestone in plasma-physics publishing.

- It remains the cleanest exposition of plasma weak-turbulence theory (and many other things).
- It should be required reading for just about everybody (certainly present and future theorists).

What has happened over the subsequent 35 years?

- Renormalized strong plasma turbulence theory (with many practical subtopics)
- Stochasticity and nonlinear dynamics as applied to plasmas
- Nonlinear gyrokinetic formalism
- Advanced computation

Plasma turbulence theory dates from the 1960's.

1959 – Kraichnan's direct-interaction approximation (DIA)

1960

61

62 – Quasilinear theory (QLT)

63 – Multiple-time-scale methods (Frieman; Sandri);
[Lorenz: Deterministic nonperiodic flow]

64 – Weak-turbulence theory (WTT)

65 – Kadomtsev's monograph

66 – Resonance-broadening theory (RBT) (Dupree; Weinstock)

67 – Orszag & Kraichnan: critique of RBT; plasma DIA

68

69 – Chirikov's thesis (on stochasticity)

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The 1970's: Ron's Book; MSR Formalism; Stochasticity

1970

71

72 – Davidson: *Methods in Nonlinear Plasma Theory*

73 – Martin–Siggia–Rose formalism (classical renormalization)

74

75 – Stochastic acceleration & heating (Smith & Kaufman; Karney)

76

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78 – Plasma DIA (DuBois; Krommes);
Treve's review: "Theory of chaotic motion with application to controlled fusion research"

79 – Littlejohn: "A guiding center Hamiltonian: A new approach"

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The Methodology of Weak Turbulence Theory

For weak turbulence, the method of multiple time scales is very well suited (see Davidson):

$$\dot{\psi} = \underbrace{L}_{(i\Omega-\gamma)} \psi + \frac{1}{2} M \psi \psi. \tag{1}$$

Cumulants: $C_1(t) \doteq \langle \psi \rangle (t)$, $C_2(t, t') \doteq \langle \delta\psi(t) \delta\psi(t') \rangle$, . . . :

$$\partial_t C_1 = L C_1 + \frac{1}{2} M (C_1^2 + C_2), \tag{2a}$$

$$\partial_t C_2 = L C_2 + M C_1 C_2 + \frac{1}{2} M C_3, \tag{2b}$$

$$\partial_t C_3 = L C_3 + M C_1 C_3 + \frac{1}{2} M (2 C_2^2 + C_4), \tag{2c}$$

$$\vdots$$

Crudely, let $\gamma/\Omega = O(\epsilon)$. Then a consistent ordering is

$$C_1 \sim 1, \quad C_2 \sim \epsilon, \quad C_3 \sim \epsilon^2, \quad C_4 \sim \epsilon^3, \quad \ldots . \tag{3}$$

Expand in multiple time scales:

$$\left(\frac{\partial}{\partial t_0}+\epsilon\frac{\partial}{\partial t_1}+\epsilon^2\frac{\partial}{\partial t_2}+\cdots\right)\left(\epsilon C_2^{(1)}+\epsilon^2 C_2^{(2)}+\cdots\right)$$
$$-2\epsilon\gamma\left(\epsilon C_2^{(1)}+\epsilon^2 C_2^{(2)}\right)$$
$$=M\left(\epsilon^2 C_3^{(2)}+\epsilon^3 C_3^{(3)}+\cdots\right), \tag{4a}$$

$$\left(\frac{\partial}{\partial t_0}+\epsilon\frac{\partial}{\partial t_1}+\cdots\right)\left(\epsilon^2 C_3^{(2)}+\epsilon^3 C_3^{(3)}\right)$$
$$+3(i\Omega-\epsilon\gamma)\left(\epsilon^2 C_3^{(2)}+\cdots\right)$$
$$=M\left[2\left(\epsilon C_2^{(1)}+\epsilon^2 C_2^{(2)}\right)^2+\epsilon^3 C_4^{(1)}+\cdots\right]. \tag{4b}$$

$O(\epsilon)$:

$$\frac{\partial C_2^{(1)}(t_0,t_1,\ldots)}{\partial t_0}=0. \tag{5}$$

$O(\epsilon^2)$:

$$\frac{\partial C_2^{(1)}}{\partial t_1} + \frac{\partial C_2^{(2)}}{\partial t_0} = 2\gamma C_2^{(1)} + M C_3^{(2)}, \tag{6a}$$

$$\frac{\partial C_3^{(2)}}{\partial t_0} + 3i\Omega C_3^{(2)} = M[C_2^{(1)}]^2. \tag{6b}$$

Note: 3Ω really means $\Omega_{\vec{k}} + \Omega_{\vec{p}} + \Omega_{\vec{q}} \equiv \Delta\Omega$ (with $\vec{k} + \vec{p} + \vec{q} = \vec{0}$).

$$C_3^{(2)} = \left(\frac{e^{-i\Delta\Omega t_0} - 1}{-i\Delta\Omega} \right) M[C_2^{(1)}]^2 \to \pi\delta(\Delta\Omega), \tag{7}$$

$$\left(\frac{\partial}{\partial t_1} - 2\gamma \right) C_2^{(1)}(t_1) + \underbrace{\frac{\partial C_2^{(2)}(t_0, t_1)}{\partial t_0}}_{\substack{\text{set to } 0 \\ \text{to avoid secularity}}} = M C_3^{(2)}(t_1). \tag{8}$$

We finally arrive (schematically) at the wave kinetic equation of WTT:

$$\boxed{\left(\frac{\partial}{\partial t_1} - 2\gamma \right) C_2 = M^2 \delta(\Delta\Omega) C_2^2.} \tag{9}$$

The multiple-time-scale expansion breaks down for strong turbulence.

In reality, the resonance is broadened:

$$\pi\delta(\Delta\Omega) \Rightarrow \mathrm{Re}\left(\frac{1}{-i(\Delta\Omega + i\Delta\eta)}\right) = \frac{\Delta\eta}{\Delta\Omega^2 + \Delta\eta^2} \tag{10a}$$

$$\approx \frac{\Delta\eta}{\Delta\Omega^2} + O(\Delta\eta^3) + O(\Delta\eta^5) + \cdots . \tag{10b}$$

Terms through ALL ORDERS are required to correctly account for the turbulent broadening. Note that perturbation theory *does not work* near the resonance $\Delta\Omega = 0$. Renormalization is required (see later discussion).

The order-by-order construction inherent in the multiple-time-scale expansion is not well suited to situations in which renormalization is essential.

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Resonances and secularities underlie the origins of stochasticity, chaos, and turbulence.

Recall Hamiltonian dynamics and the problem of resonant denominators:

$$H(\vec{J},\vec{\theta}) = H_0(\vec{J}) + \delta H(\vec{J},\vec{\theta}). \tag{11}$$

Equations of motion:

$$\frac{\partial H}{\partial \vec{J}} = \dot{\vec{\theta}} = \vec{\Omega}(\vec{J}) + \frac{\partial \delta H}{\partial \vec{J}}, \tag{12a}$$

$$\frac{\partial H}{\partial \vec{\theta}} = -\dot{\vec{J}} = \frac{\partial \delta H}{\partial \vec{\theta}}. \tag{12b}$$

Let $\delta H = \sum_{\vec{m}} e^{i\vec{m}\cdot\vec{\theta}} H_{\vec{m}}$. Then with $\vec{\theta} \approx \vec{\theta}_0 + \vec{\Omega} t$,

$$\Delta \vec{J} = -\int dt\, \frac{\partial \delta H}{\partial \vec{\theta}} \approx -i \sum_{\vec{m}} \frac{\vec{m} H_{\vec{m}} e^{i\vec{m}\cdot\vec{\theta}_0}(e^{i\vec{m}\cdot\vec{\Omega}t} - 1)}{\vec{m}\cdot\vec{\Omega}}. \tag{13}$$

Trouble near $\vec{m}\cdot\vec{\Omega}(\vec{J}) = 0$ (see Kolmogorov–Arnold–Moser).

Analogous problems occur in forced, dissipative systems; cf. strange attractors.

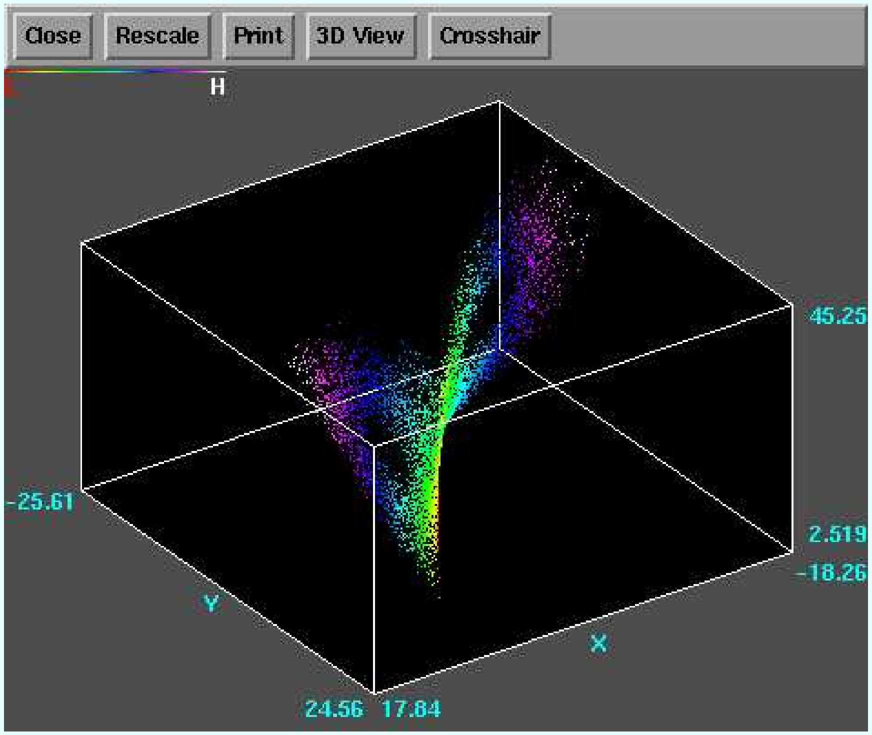

Fig. 1. The Lorenz strange attractor.

"butterfly effect" ⇒

- statistical description;
- cumulant hierarchy *through all orders*;
- renormalization.

Stochastic / chaotic / turbulent systems exhibit exponential sensitivity to small changes in initial conditions.

A Very Brief Introduction to Renormalization

A model of random passive advection illustrates the basic points:

$$\partial_t \psi(\vec{x}, t) + \tilde{\vec{V}}(\cancel{\vec{x}}, t) \cdot \vec{\nabla} \psi = 0, \tag{14a}$$

$$\partial_t \psi_{\vec{k}}(t) + i\tilde{\Omega}(t)\psi_{\vec{k}} = 0, \tag{14b}$$

where $\tilde{\Omega}(t) \doteq \vec{k} \cdot \tilde{\vec{V}}(t)$. Further, let $\tilde{\Omega}(t) \to \tilde{\Omega}$, a Gaussian random number with $\langle \tilde{\Omega} \rangle = 0$, $\langle \tilde{\Omega}^2 \rangle = \beta^2$. Thus, study

$$\partial_t \psi = -i\tilde{\Omega}\psi. \tag{15}$$

The average response of the oscillator is characterized by the *mean Green's function* (*infinitesimal response function*) $R(t; t')$, which obeys in general

$$\partial_t R(t; t') + \int_{t'}^{t} d\bar{t} \quad \underbrace{\Sigma(t; \bar{t})}_{\substack{\text{memory fcn.} \\ \text{(turbulent coll op.)}}} \quad R(\bar{t}; t') = \delta(t - t'). \tag{16}$$

The goal is to calculate the memory function Σ.

Linear dissipation is required to regularize an infinity.

Less ambitiously, characterize the oscillator by a Markovian approx.,

$$\partial_\tau R(\tau) + \eta^{\mathrm{nl}} R = \delta(\tau), \tag{17}$$

where $\eta^{\mathrm{nl}} \doteq \int_0^\infty d\tau\, \Sigma(\tau)$, and try to calculate η^{nl}. In lowest-order perturbation theory (quasilinear level), it is easy to find that

$$\Sigma^{\mathrm{QL}}(\tau) = \beta^2 R^{(0)}(\tau) \quad \text{(causal, but otherwise a constant)}, \tag{18}$$

where $R^{(0)}(\tau) = H(\tau)$ (unit step function). Thus

$$\eta^{\mathrm{QL}} = \int_0^\infty d\tau\, \beta^2 \underbrace{R^{(0)}(\tau)}_{1} = \begin{cases} 0 & (\beta = 0), \\ \infty & (\beta \neq 0). \end{cases} \tag{19}$$

To regularize the infinity, add some linear damping:

$$\partial_t \psi + \nu\psi = -i\widetilde{\Omega}\psi. \tag{20}$$

Now $R^{(0)}(\tau) \to H(\tau)e^{-\nu\tau}$, and $\eta^{\mathrm{QL}} = \int_0^\infty d\tau\, \Sigma^{\mathrm{QL}}(\rho) = \beta^2/\nu$ (a continuous function of β).

Summing through all orders amounts to enforcing self-consistency — the essence of renormalization.

With perturbation theory regularized, one can now proceed order by order, and eventually sum through all orders. The result is the replacement $R^{(0)} \to R$, i.e.,

$$\Sigma^{\mathrm{nl}}(\tau) = \beta^2 R(\tau), \tag{21}$$

where $R(\tau) = H(\tau)\exp[-(\nu + \eta^{\mathrm{nl}})\tau]$. Thus we obtain the self-consistent equation

$$\eta^{\mathrm{nl}} = \frac{\beta^2}{\nu + \eta^{\mathrm{nl}}}, \tag{22}$$

which can be solved for η^{nl}. In particular, one may now take the limit $\nu \to 0$, whereupon one finds $\eta^{\mathrm{nl}} = \beta$. Note the appearance of *anomalous scaling*:

$$\eta^{\mathrm{nl}} = \begin{cases} \beta^2/\nu & \text{(quasilinear theory),} \\ \beta^1 & \text{(renormalized).} \end{cases} \tag{23}$$

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Progress in Statistical Closure Theory

1980

81

82 – Terry & Horton: Drift-wave stochasticity;
Krommes: DIA for three-mode Terry–Horton equations

83 – Waltz: Numerical study of a Markovian closure;
[Lee: Gyrokinetic approach to particle simulation];
[Dubin[a] et al.: Hamiltonian approach to nonlinear gyrokinetic equations]

⋮ – More DIA and other closure calculations;
fluctuation noise in PIC simulations;
variational methods for bounding turbulent fluxes (Smith; Kim)

~1990 – A major surprise and embarrassment: EDQNM closure doesn't work with waves (Bowman)

[a] Green ≡ Princeton graduate student.

Progress, continued

1993 – Bowman et al.: Realizable Markovian Closure (RMC)

94

95 – Hu et al.: Hasegawa–Wakatani equations and the RMC

96 – [Holmes, Lumley, & Berkooz: *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*]

97

98 – Diamond et al.: Zonal flows and self-regulating DW turbulence

99

2000 – Krommes & Kim: Interactions of disparate scales in drift-wave turbulence (unification of theories of ZFs and eddy viscosity)

⋮

2005 – Nevins, Hammett, et al.: Sampling noise in PIC simulations; Kolesnikov: "Bifurcation theory of the transition to collisionless ion-temperature-gradient-driven plasma turbulence"

⋮

An Example of State-of-the-Art Research on Plasma Turbulence: Blob Generation in a Turbulent Tokamak Edge

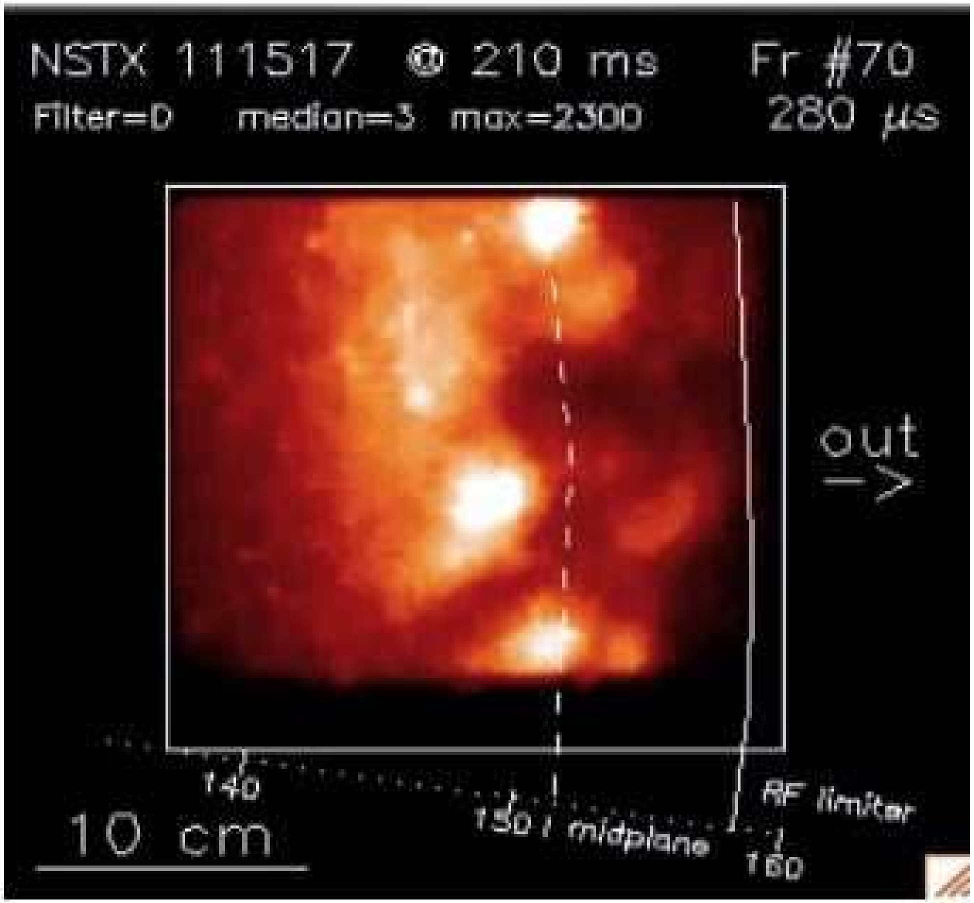

Fig. 2. Visualization of edge turbulence in NSTX using the gas-puff-imaging diagnostic. See `http://www.pppl.gov/~szweben/` **for movies and more.**

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The theory of blob generation requires knowledge and synthesis of many facets of linear and nonlinear physics.

In the spirit of *Methods of Nonlinear Plasma Theory*, we would like to study blob generation analytically. This requires understanding of

- experimental phenomenology
- linear modes in a tokamak edge
- properties of turbulent saturated states
- nonlinear dynamics
- intermittency
- formation of coherent structures

Analytical theories of the red topics are not well developed.

A research program to study blob generation theoretically is extremely challenging.

- Derive paradigmatic nonlinear equation for NSTX edge.
- Numerically simulate to determine principal orthogonal eigenfunctions (good basis functions for the inhomogeneous turbulence).
- Galerkin project to deduce manageable system of coupled ODEs.
- Qualitative nonlinear-dynamics analysis (cycles in phase space describe birth–propagation–destruction of blobs).
- Estimate of blob generation rate
  - Understand dependence on important physical parameters.
  - Compare with experiment and simulations.
- GRADUATE with PhD (Stoltzfus-Dueck)!

The ultimate goal is *dynamical control*.

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To better train the next generation of plasma physicists, we need to rethink our curricula.

Note: *Append "for plasmas" to all topics below.*

For everyone:

- more focus on nonlinear dynamics (appreciate the richness of possible behavior)
- fractal Brownian motion
- basics of fluid turbulence theory
- gyrokinetics (de-emphasize Braginskii)

For the hard-core theorists:

- Hamiltonian–Lie methods
- bifurcation theory
- serious dynamical-systems analysis
- renormalization techniques
- PDF methods
- intermittency
- coherent structures

These would be good chapters for Ron's next book, *Methods in Nonlinear Plasma Theory, Vol. II*.

Summary

The organization of Ron's 1972 book applies just as well today, and highlights a challenging list of state-of-the-art problems in nonlinear plasma physics:

- Part I. Coherent Nonlinear Phenomena
  - $\vec{E} \times \vec{B}$ **trapping**
  - **blob generation**
- Part II. Turbulent Nonlinear Phenomena
  - **zonal flows**
  - **anisotropic cascades in magnetic turbulence**

Ron's pioneering research and effective exposition have inspired multiple generations of scientists. He is a great role model.

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References

Bowman, J. C., Krommes, J. A., and Ottaviani, M. (1993). The realizable Markovian closure. I. General theory, with application to three-wave dynamics. Phys. Fluids B, 5:3558–3589.

Chirikov, B. V. (1969). Research concerning the theory of non-linear resonance and stochasticity. Nuclear Physics Institute of the Siberian Section of the USSR Academy of Sciences Report 267, translated by A. T. Sanders (CERN Trans. 71–40).

Davidson, R. C. (1972). Methods in Nonlinear Plasma Theory. Academic Press, New York.

Diamond, P. H., Rosenbluth, M. N., Hinton, F. L., Malkov, M., Fleischer, J., and Smolyakov, A. (1998). Dynamics of zonal flows and self-regulating drift-wave turbulence. In 17th IAEA Fusion Energy Conference, pages 1421–1428. International Atomic Energy Agency, Vienna. IAEA–CN–69/TH3/1.

Drummond, W. E. and Pines, D. (1962). Non-linear stability of plasma oscillations. In Proceedings of the Conference on Plasma Physics and

Controlled Nuclear Fusion Research (Salzburg, 1961) [Nucl. Fusion Suppl. Pt. 3], pages 1049–1057. International Atomic Energy Agency, Vienna.

Dubin, D. H. E., Krommes, J. A., Oberman, C. R., and Lee, W. W. (1983). Nonlinear gyrokinetic equations. Phys. Fluids, 26:3524–3535.

DuBois, D. F. and Espedal, M. (1978). Direct interaction approximation and plasma turbulence theory. Plasma Phys., 20:1209–1239.

Dupree, T. H. (1966). A perturbation theory for strong plasma turbulence. Phys. Fluids, 9:1773–1782.

Dupree, T. H. (1967). Nonlinear theory of drift-wave turbulence and enhanced diffusion. Phys. Fluids, 10:1049–1055.

Frieman, E. A. (1963). On a new method in the theory of irreversible processes. J. Math. Phys., 4:410.

Holmes, P., Lumley, J. L., and Berkooz, G. (1996). Turbulence, Coherent Structures, Dynamical Systems and Symmetry. Cambridge University Press, Cambridge.

Hu, G., Krommes, J. A., and Bowman, J. C. (1995). Resistive drift-wave plasma turbulence and the realizable Markovian closure. Phys. Lett. A, 202:117–125.

Hu, G., Krommes, J. A., and Bowman, J. C. (1997). Statistical theory of resistive drift-wave turbulence and transport. Phys. Plasmas, 4:2116–2133.

Kadomtsev, B. B. (1965). Plasma Turbulence. Academic Press, New York. Translated by L. C. Ronson from the 1964 Russian edition *Problems in Plasma Theory*, edited by M. A. Leontovich. Translation edited by M. C. Rusbridge.

Kolesnikov, R. A. and Krommes, J. A. (2005). Bifurcation theory of the transition to collisionless ion-temperature-gradient-driven plasma turbulence. Phys. Plasmas, 12:122302.

Kraichnan, R. H. (1959). The structure of isotropic turbulence at very high Reynolds numbers. J. Fluid Mech., 5:497–543.

Krommes, J. A. (1978). Turbulence, clumps, and the Bethe–Salpeter equation. In Theoretical and Computational Plasma Physics, pages 405–417. International Atomic Energy Agency, Vienna.

Krommes, J. A. (1982). The direct-interaction approximation and statistically steady states of three nonlinearly coupled modes. Phys. Fluids, 25:1393–1395.

Krommes, J. A. and Kim, C.-B. (2000). Interactions of disparate scales in drift-wave turbulence. Phys. Rev. E, 62:8508–8539.

Lee, W. W. (1983). Gyrokinetic approach in particle simulation. Phys. Fluids, 26:556–562.

Littlejohn, R. G. (1979). A guiding center Hamiltonian: A new approach. J. Math. Phys., 20:2445–2458.

Lorenz, E. N. (1963). Deterministic nonperiodic flow. J. Atmos. Sci., 20:130.

Martin, P. C., Siggia, E. D., and Rose, H. A. (1973). Statistical dynamics of classical systems. Phys. Rev. A, 8:423–437.

Nevins, W. M., Hammett, G. W., Dimits, A. M., Dorland, W., and Shumaker, D. E. (2005). Discrete particle noise in particle-in-cell simulations of plasma microturbulence. Phys. Plasmas, 12:122305.

Orszag, S. A. and Kraichnan, R. H. (1967). Model equations for strong turbulence in a Vlasov plasma. Phys. Fluids, 10:1720–1736.

Sandri, G. (1965). New method of expansion in mathematical physics — I. Nuovo Cimenta, 36:67.

Smith, G. R. and Kaufman, A. N. (1975). Stochastic acceleration by a single wave in a magnetic field. Phys. Rev. Lett., 34:1613.

Terry, P. and Horton, W. (1982). Stochasticity and the random phase approximation for three electron drift waves. Phys. Fluids, 25:491.

Treve, Y. M. (1978). Theory of chaotic motion with application to controlled fusion research. In Jorna, S., editor, Topics in Nonlinear Dynamics: A Tribute to Sir Edward Bullard, page 147, New York. AIP.

Vedenov, A., Velikhov, E., and Sagdeev, R. (1962). The quasi-linear theory of plasma oscillations. In Proceedings of the Conference on Plasma Physics and Controlled Nuclear Fusion Research (Salzburg, 1961) [Nucl. Fusion Suppl. Pt. 2], pages 465–475. International Atomic Energy Agency, Vienna. Translated in U.S.A.E.C. Division of Technical Information document AEC–tr–5589 (1963), pp. 204–237.

Waltz, R. E. (1983). Numerical study of drift wave turbulence with simple models for wave–wave coupling. Phys. Fluids, 26:169.

Weinstock, J. (1969). Formulation of a statistical theory of strong plasma turbulence. Phys. Fluids, 12:1045.