Geometrically Generalized Vlasov-Maxwell Equations and Applications to Gyrokinetic Theories and Algorithms

-- Symmetry, Gauge, Kruskal Ring, and Data Structure

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http://www.pppl.gov/~hongqin/Gyrokinetics.php

Motivation – develop good approximations and algorithms

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + e_j\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right] f_j(\boldsymbol{x},\boldsymbol{p},t) = 0$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_j e_j \int d^3p\,\boldsymbol{v} f_j(\boldsymbol{x},\boldsymbol{p},t) + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_j e_j \int d^3p\, f_j(\boldsymbol{x},\boldsymbol{p},t)$$

$$\nabla\cdot\boldsymbol{B} = 0$$

$\longleftrightarrow$ Approximations (bad/good)

$\downarrow$

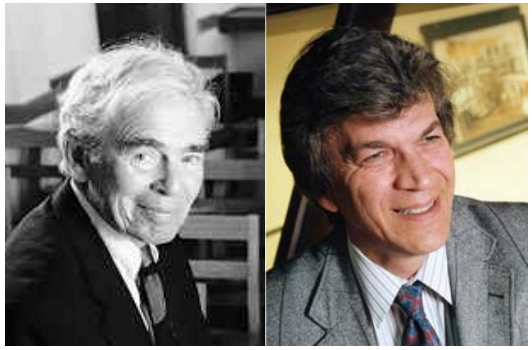
Advance in Plasma Physics

Good approximations:

- Phase space conservation
- Energy-momentum conservation
- Enable efficient and conservative algorithms

Example: gyrokinetic approximaiton and algorithm -- average out gyrophase

Frieman, Davidson, Langdon (1966)



- Magnetized plasmas $\rightarrow$ fast gyromotion.
- "Average-out" the fast gyromotion
  - Theoretically appealing
  - Algorithmically efficient

Highly oscillatory

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+e_j\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial}{\partial \mathbf{p}}\right]f_j(\mathbf{x},\mathbf{p},t)=0$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\sum_j e_j\int d^3p\,\mathbf{v}f_j(\mathbf{x},\mathbf{p},t)+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}$$

$$\nabla\cdot\mathbf{E}=4\pi\sum_j e_j\int d^3p\,f_j(\mathbf{x},\mathbf{p},t)$$

$$\nabla\cdot\mathbf{B}=0$$

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\,[\text{ Valsov}-\text{Maxwell eqs. }]$$

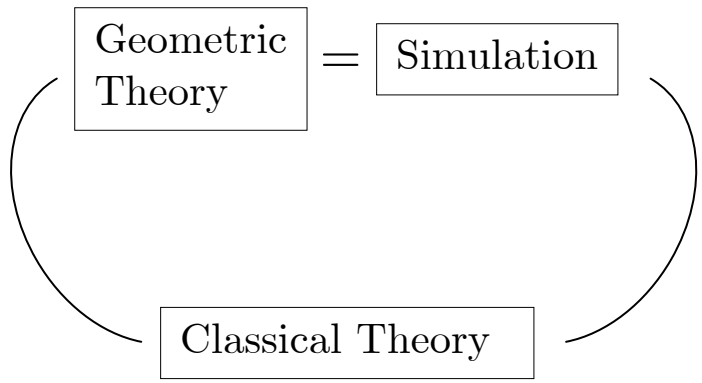
θ dependent

$$\left\langle \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}\right\rangle,\ \left\langle \frac{\mathbf{v}\times\mathbf{B}}{c}\cdot\frac{\partial f}{\partial \mathbf{p}}\right\rangle?$$

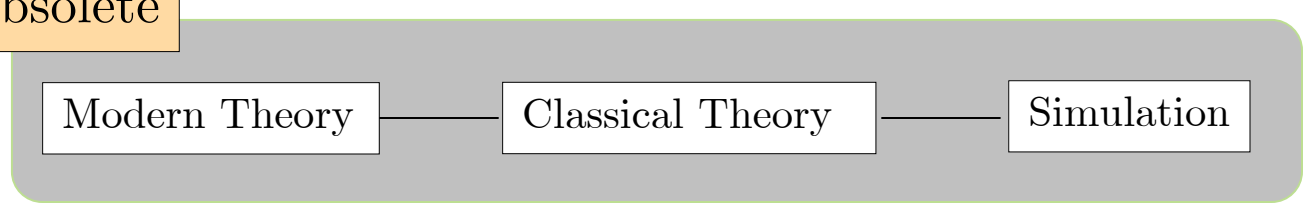
Conservation properties?

Solution – geometrically generalized Vlasov-Maxwell equations

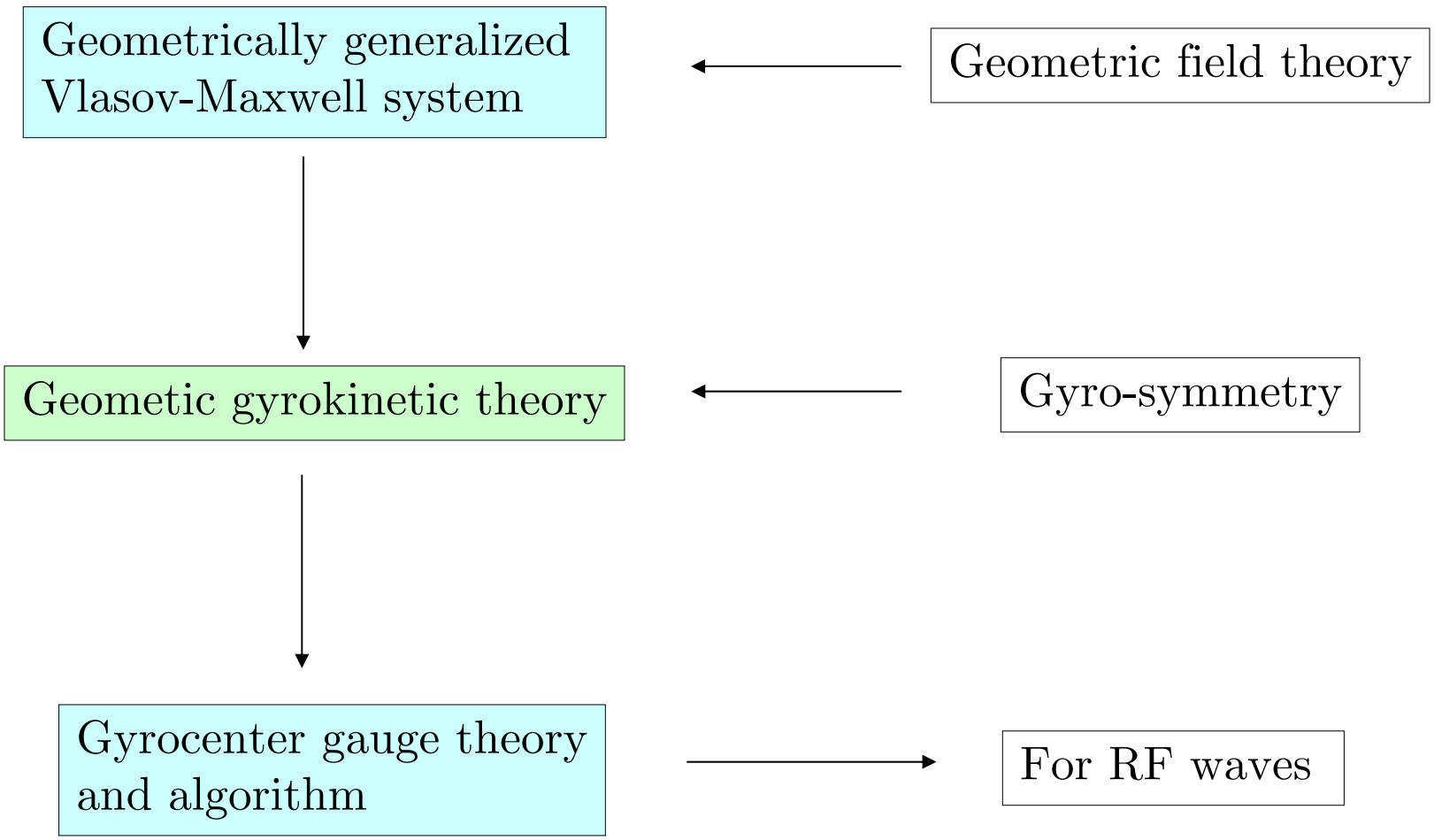
Geometry = Fundamental theory = Good algorithm



Obsolete



Outline


Geometrically generalized Vlasov-Maxwell system
Geometric field theory
Geometic gyrokinetic theory
Gyro-symmetry
Gyrocenter gauge theory and algorithm
For RF waves

Outline of physics applications

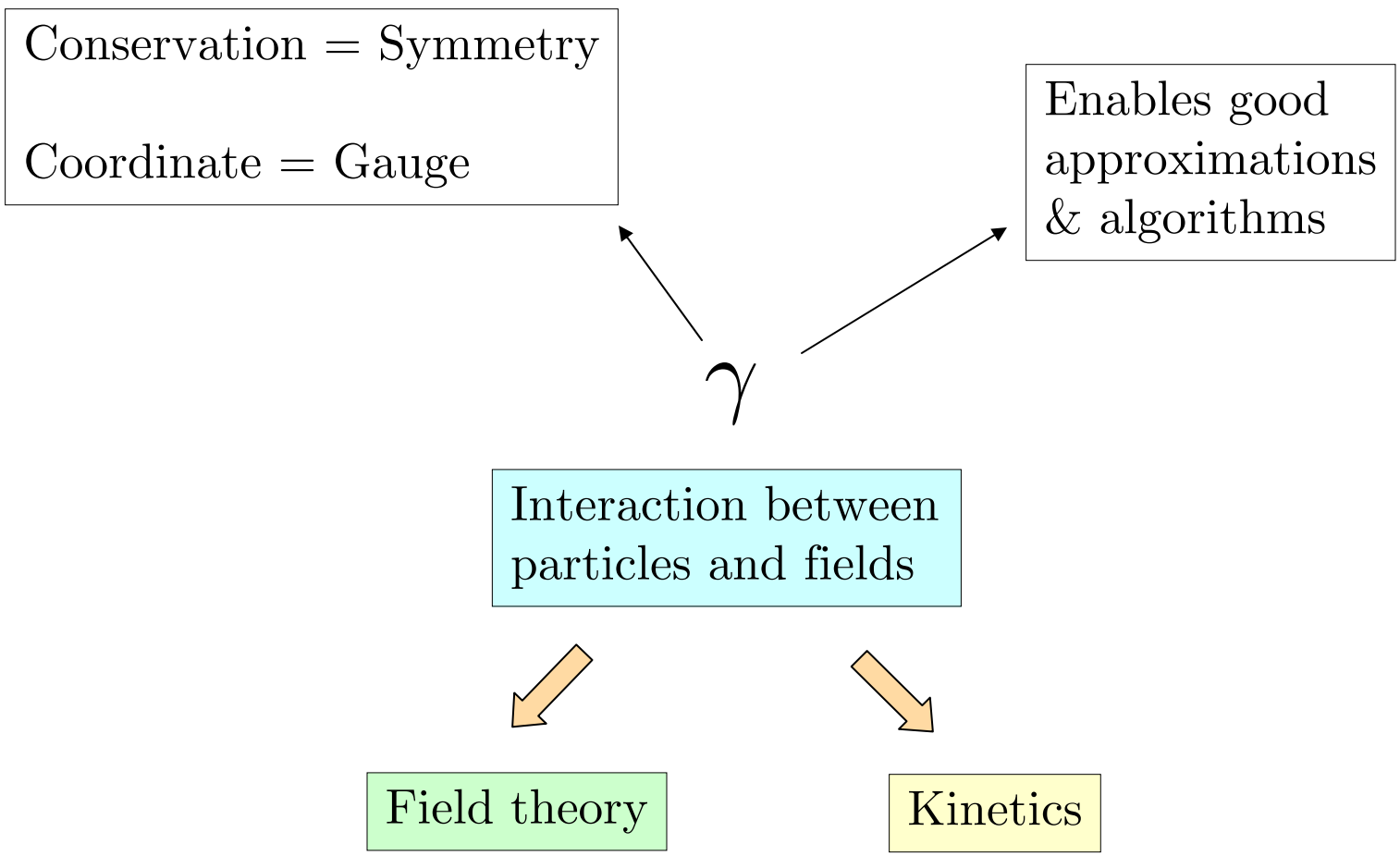
Geometric field theory

- Diamagnetic current
- Polarization density

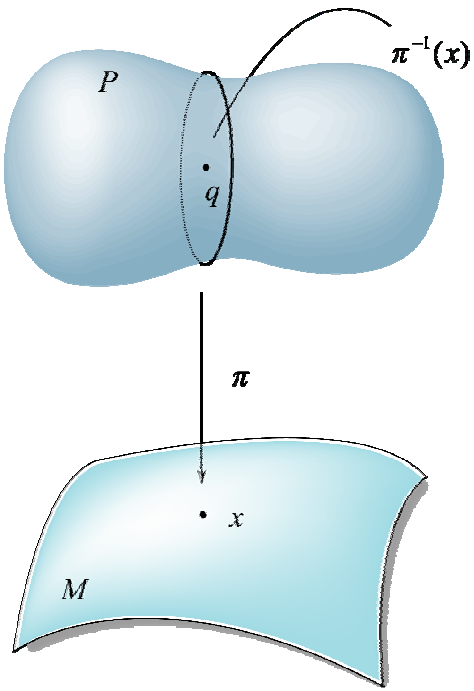
Gyrocenter gauge theory

- Complete plasma susceptibility
- Gyrocenter gauge algorithm
- Kruskal ring data structure
- Stochastic resonant heating

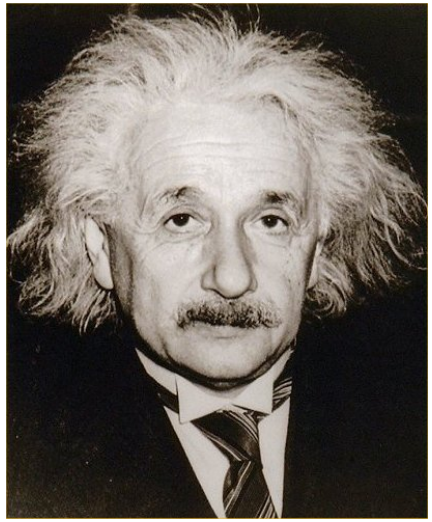
Geometric field theory from Poincare-Cartan-Einstein 1-form γ



Where does γ live? -- phase space



Phase space is a fiber bundle $\pi : P \longrightarrow M$.



Cotagent bundle

$$P = \{(x,p) \,|\, x \in M,\, p \in T_x^* M,\, g^{-1}(p,p) = -m^2c^2\}$$

7D Spacetime

Inverse of metric

Poincare-Cartan-Einstein 1-form $\gamma \rightarrow$ particle dynamics

Hamilton's Eq.

Inner product

Exterior derivative

$$i_\tau d\gamma = 0$$

Worldline

Different for different particles: charged particle, neutron, polymer, virus, ...

Geometrically generalized Vlasov-Maxwell system --- A field theory

Liouville 6-form

$$\Omega \equiv -\frac{1}{6} d\gamma \wedge d\gamma \wedge d\gamma \,,$$

$$L_\tau \Omega = i_\tau d\Omega + d\big(i_\tau \Omega\big) = 0.$$

Liouville Theorem

Vlasov Eq.

$$L_\tau f \;=\; i_\tau df \;=\; 0.$$

$$L_\tau\,(f\Omega) = (L_\tau f)\Omega + (L_\tau \Omega) f \;=\; 0.$$

Conservative form

Geometrically generalized Vlasov-Maxwell system --- A field theory

$$S = \int_x L,$$

$$L(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

Variational derivative

$$\frac{\delta S}{\delta A} = E(L) = 0$$

Fiber integral

Euler derivative

$$d * dA = 4\pi * j,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\delta\gamma(x')}{\delta A_{\alpha}(x)}, (\alpha = 0,1,2,3).$$

Second order field theory → 4-diamagnetic current

$$j^{\alpha}(x) = \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha}(x)}$$
$$-\frac{\partial}{\partial x^{\beta}}\left[\int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha,\beta}(x)}\right], (\alpha = 0,1,2,3).$$

4-diamagnetic current

$$\frac{\partial A_{\alpha}}{\partial x^{\beta}}$$

Valid for any γ. Exact conservation properties.

Allow physics models, approximations build into γ.

Example: 1-form for charge particles with Lorentz force

$$\gamma = A + p = (\boldsymbol{A} + \boldsymbol{v}) \cdot d\boldsymbol{x} - \left[\frac{v^2}{2} + \phi \right] dt \,,$$

$A \equiv (-\phi, \boldsymbol{A})$, four vector potential, 1-form

$p \equiv (-v^2/2, \boldsymbol{p})$, 1-form momentum

$\boxed{\times dt}$

$$L = \boldsymbol{A} \cdot \boldsymbol{v} + \frac{1}{2} v^2 - \phi$$

$$= (\boldsymbol{A} + \boldsymbol{v}) \cdot \boldsymbol{v} - \left(\frac{1}{2} v^2 + \phi \right)$$

Action

$$L(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

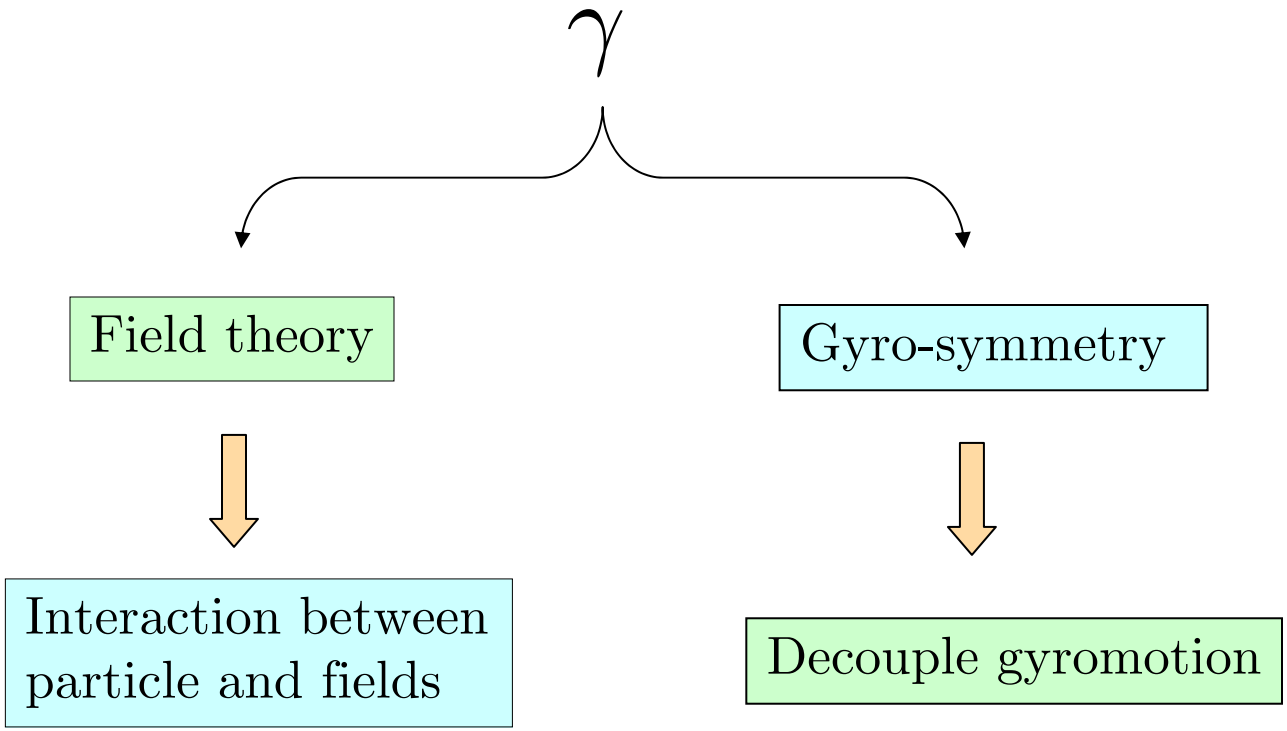
Vlasov

$$df(\tau) = 0, \; i_\tau d\gamma = 0$$

Maxwell

$$d * dA = 4\pi \int_{\pi^{-1}(x)} f\Omega$$

Modern gyrokinetics = field theory + gyro-symmetry



What is symmetry?

- Coordinate dependent version:

$$\frac{\partial L}{\partial \theta} = 0\,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0\,.$$

Problem, what is θ?

- Geometric version:

S. Lie (1890s)

Symmetry is group

$$L_\eta \gamma = dS$$

Lie derivative

Symmetry vector field

Advantage: general, stronger, enables techniques to find θ.

Symmetry is invaraint

Noether's Theorem (1918)

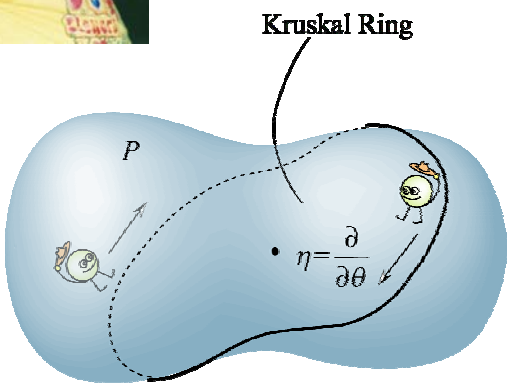
Cartan's formula

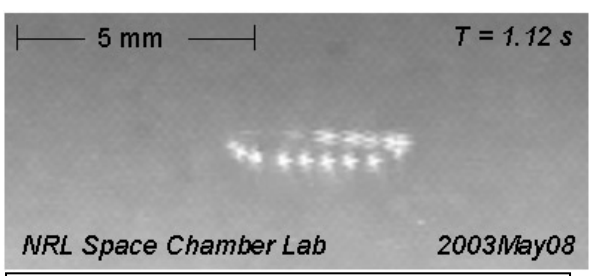
$$L_\eta \gamma = d\left(i_\eta \gamma\right) + i_\eta d\gamma = dS$$

$$d(\gamma \cdot \eta) \cdot \tau = dS \cdot \tau$$

$\gamma \cdot \eta - S$ is conversed.

What is gyrosymmetry?





Amatucci, Pop 11, 2097 (2004).

$$\eta = v_x \left(\frac{1}{B}\frac{\partial}{\partial x} + \frac{\partial}{\partial v_y} \right) + v_y \left(\frac{1}{B}\frac{\partial}{\partial y} - \frac{\partial}{\partial v_x} \right)$$

$$\eta = \frac{\partial}{\partial \theta}$$

Noether's Theorem

$$\mu = \frac{v_x^2 + v_y^2}{2B}$$

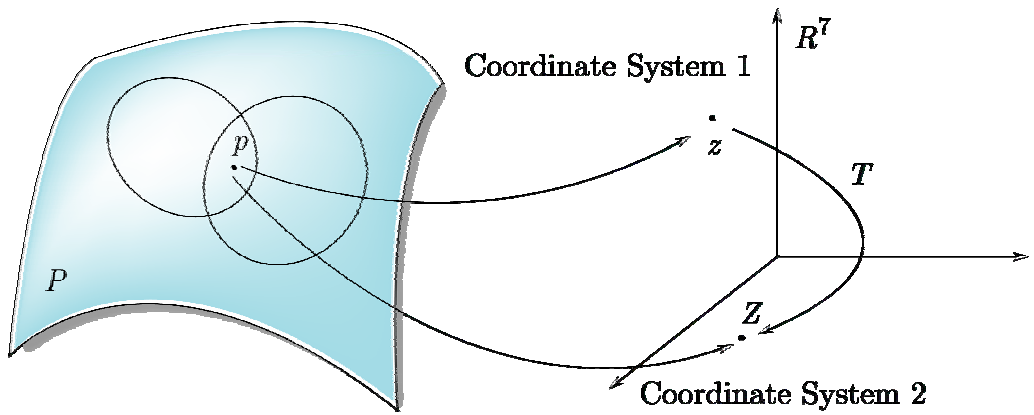
Gyrophase coordinate

Coordinate perturbation to find η.

Q: How does the dynamics transform?

A: It transforms as an 1-form!

Dynamics under Lie group of coordinate transformation



Continuous Lie group,

$$g : z \mapsto Z = g(z, \varepsilon)$$

Vector field (Lie algbra),

$$G = dg/d\varepsilon\,|_{\varepsilon=0} \quad ,$$

$$-G = dg^{-1}/d\varepsilon\,|_{\varepsilon=0} \; .$$

γ in Z

Pullback

Taylor expansion

$$\Gamma(Z) = g^{-1*}\gamma(z) = \gamma\left[g^{-1}(Z)\right] = \gamma(Z) - L_{G(Z)}\gamma(Z) + O(\varepsilon^2)$$

$$= \gamma(Z) - i_{G(Z)}d\gamma(Z) - d\left[\gamma \cdot G(Z)\right] + O(\varepsilon^2) \, .$$

Cartan's formula

Insignificant

No need for Poisson bracket

Gyrocenter dynamics

$$i_\tau d\gamma = 0\,.$$

Curvature drift

$\boldsymbol{E}\times\boldsymbol{B}$, ∇B drift

∇D & curvature $\boldsymbol{D}$ drift

drift by spacetime inhomogeneties of $\boldsymbol{E_0}$

Banos drift

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\left(u + \frac{\mu}{2}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}\right) - \frac{\boldsymbol{b}\times\boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{du}{dt} = \frac{\boldsymbol{B}^\dagger\cdot\boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{d\theta}{dt} = B_0 + \boldsymbol{R}\cdot\frac{d\boldsymbol{X}}{dt} - R_0 + \frac{\boldsymbol{E}_0\cdot\boldsymbol{\nabla}B_0}{B_0^2} + \frac{u}{2}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}$$

$$-\frac{1}{2B_0}\left[\boldsymbol{\nabla}\cdot\boldsymbol{E}_0 - \boldsymbol{b}\boldsymbol{b}:\boldsymbol{\nabla}\boldsymbol{E}_0\right] + \frac{\partial}{\partial\mu}\langle\psi_1+\psi_2\rangle$$

$$\frac{d\mu}{dt} = 0$$

$$\boldsymbol{B}^\dagger \equiv \boldsymbol{\nabla}\times(\boldsymbol{A}_0 + u\boldsymbol{b} + \boldsymbol{D}),\ B_\parallel^\dagger = \boldsymbol{B}^\dagger\cdot\boldsymbol{b}$$

$$\boldsymbol{E}^\dagger \equiv \boldsymbol{E}_0 - \boldsymbol{\nabla}\left[\mu B_0 + \frac{D^2}{2} + \langle\psi_1+\psi_2\rangle\right] - u\frac{\partial\boldsymbol{b}}{\partial t} - \frac{\partial\boldsymbol{D}}{\partial t}$$

Gyrokinetic equations

$$L_\tau F = i_\tau dF = 0.$$

$$\frac{dZ_j}{dt}\frac{\partial F}{\partial Z_j} = 0\,,\left(0 \le j \le 6\right).$$

$$F = \langle F \rangle.$$

$$\frac{\partial}{\partial \theta}\left(\frac{dZ}{dt}\right) = 0\,,$$

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot\nabla_{\mathbf{x}}\langle F \rangle + \frac{du}{dt}\frac{\partial \langle F \rangle}{\partial u} = 0\,,$$

Gyrokinetic field theory

$$\boxed{\Omega \equiv -\frac{1}{6} d\gamma \wedge d\gamma \wedge d\gamma,}$$

$$\gamma$$

$$\boxed{\begin{aligned}
&\Omega = B_{\|}^{\dagger} dx^1 \wedge dx^2 \wedge dx^3 \wedge du \wedge d\mu \wedge d\theta \\
&+\Big[A_{j,t}^{\dagger} b_i - A_{i,j}^{\dagger} u - b_j H_{,i} \Big] dt \wedge dx^j \wedge dx^i \wedge du \wedge d\mu \wedge d\theta \\
&+\Big[A_{i,j}^{\dagger} H_{,l} + A_{i,j}^{\dagger} A_{l,t}^{\dagger} \Big] dx^j \wedge dx^i \wedge dt \wedge d\mu \wedge d\theta \\
&-A_{i,j}^{\dagger} b_l H_{,\mu} dx^j \wedge dx^i \wedge dt \wedge du \wedge d\mu.
\end{aligned}}$$

$$d * dA = 4\pi * j,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\delta\gamma(x')}{\delta A_{\alpha}(x)}, (\alpha = 0,1,2,3).$$

Electromagnetic gyrokinetic system

Model

$$\gamma = \boldsymbol{A}^{\dagger} \cdot d\boldsymbol{X} + \mu d\theta - H dt,$$
$$H = \frac{u^2 + w^2 + D^2}{2} + \phi_0 .$$

Gyrocenter current

$$\boldsymbol{j} = \boldsymbol{j}_g + \boldsymbol{j}_M ,$$

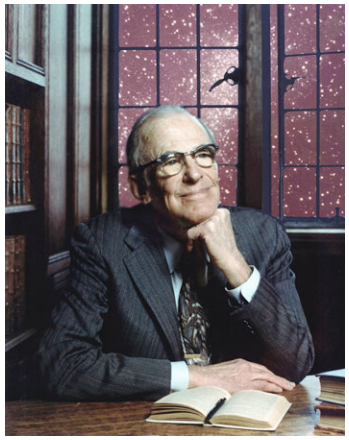
$$\boldsymbol{j}_g = \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\partial \gamma(x')}{\partial A_i(x)}$$
$$= \int F \left(\frac{\boldsymbol{E}^{\dagger} \times \boldsymbol{b} + \boldsymbol{B}^{\dagger} u}{B_{\parallel}^{\dagger}} \right) B_{\parallel}^{\dagger} du \wedge d\mu \wedge d\theta .$$

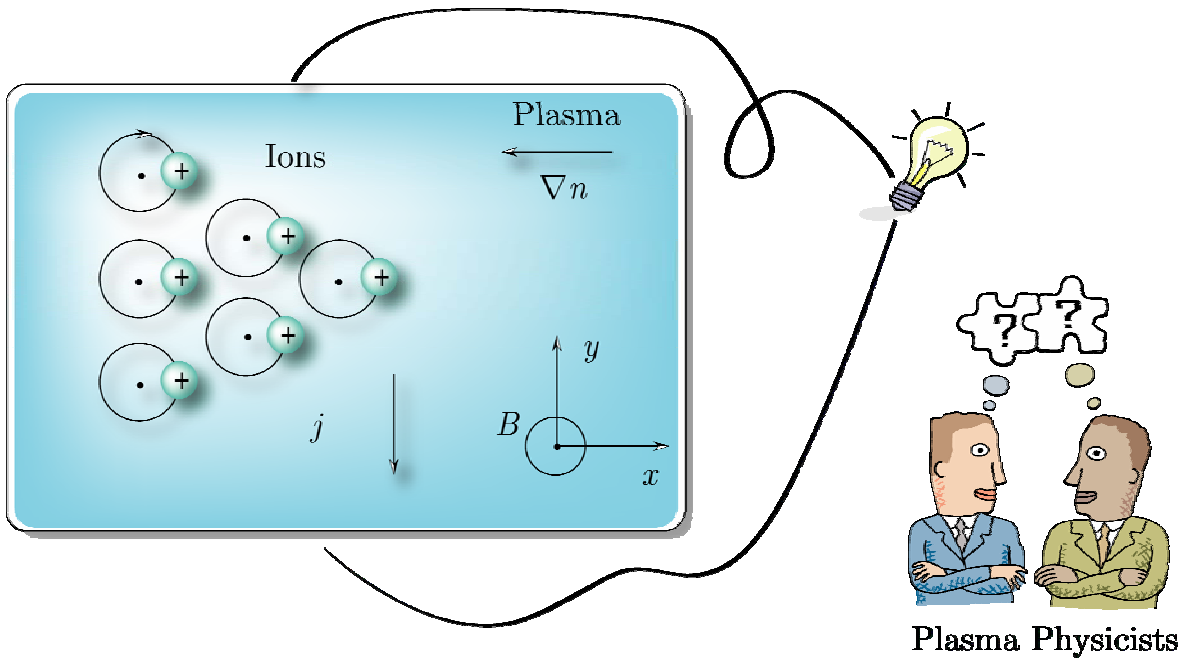
All drifts except Banos

Dia-magnetic current

$$\boldsymbol{j}_M = -\frac{\partial}{\partial x^j} \left[\int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{i,j}(x)} \right]$$
$$= -\nabla \times \left[\boldsymbol{b} \int \mu F B_{\parallel}^{\dagger} du \wedge d\mu \wedge d\theta \right].$$

Physics of 2nd order field theory — Spitzer paradox





$$\boxed{\boldsymbol{j}_y = \boldsymbol{j}_g + \boldsymbol{j}_M = \left(\boldsymbol{b} \times \frac{\nabla p}{B} \right)_y}$$

0

Electrostatic gyrokinetic system

$$\gamma = \boldsymbol{A}^{\dagger} \cdot d\boldsymbol{X} + \mu d\theta - H dt,$$
$$H = \frac{u^2 + w^2 + D^2}{2} + \phi_0 + \langle \psi_1 \rangle.$$

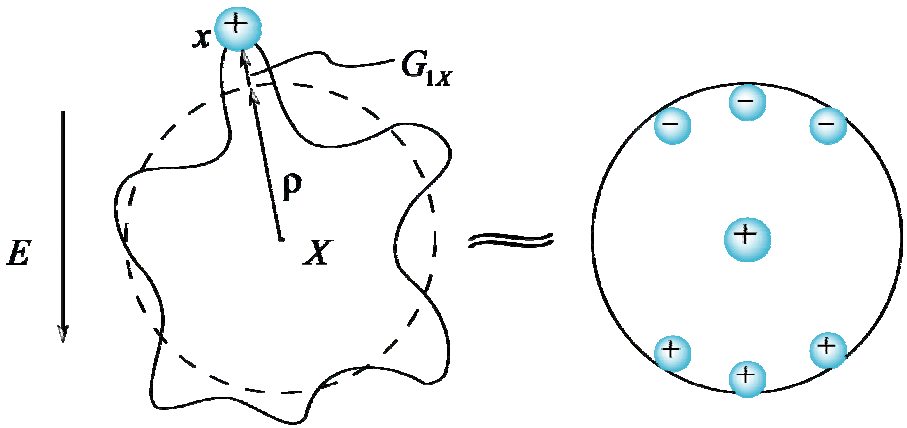
- Exact energy conservation
- Inhomogeneous EM background

$$\langle \psi_1 \rangle = \langle \phi_1(\boldsymbol{X} + \boldsymbol{\rho}) \rangle$$
$$s_1 = \frac{1}{B_0} \int \tilde{\phi}_1(\boldsymbol{X} + \boldsymbol{\rho}) d\theta \equiv \frac{1}{B_0} \tilde{\phi}_1^{(1)},$$
$$\langle \psi_2 \rangle = -\frac{1}{2} \left\langle \nabla \tilde{\phi}_1 \cdot \nabla \tilde{\phi}_1^{(1)} \right\rangle \times \frac{\boldsymbol{b}}{B_{\parallel}^{\dagger} \bar{B}_0} - \frac{1}{2B_0} \left\langle \frac{\partial \tilde{\phi}_1^2}{\partial \mu} \right\rangle,$$

$$n(x) = \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \left[\frac{\partial \langle \psi_1 \rangle}{\partial \phi_1(x)} + \frac{\partial \langle \psi_2 \rangle}{\partial \phi_1(x)} \right]$$
$$= \int \left[F + \frac{\partial F}{\partial \mu} \frac{\tilde{\phi}_1}{B_0} + \nabla F \times \frac{\nabla \tilde{\phi}_1^{(1)}}{B_{\parallel}^{\dagger} \bar{B}_0} \cdot \boldsymbol{b} + \frac{\partial F}{\partial u} \frac{\boldsymbol{B}^{\dagger} \cdot \nabla \tilde{\phi}_1^{(1)}}{B_{\parallel}^{\dagger} \bar{B}_0} \right]$$
$$\times \delta\left[\boldsymbol{x} - \boldsymbol{x}' - \boldsymbol{\rho}(\boldsymbol{x}') \right] B_{\parallel}^{\dagger} d^3x du d\mu d\theta \,.$$

Pullback transformation

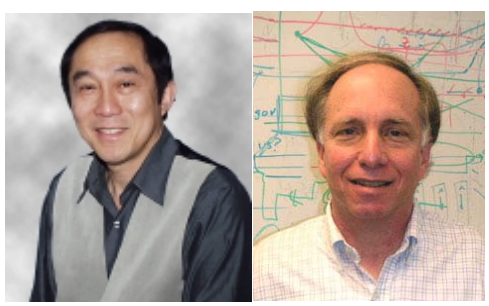
Physics of 2nd order field theory — polarization density



$$x = X + \rho + G_{1X} \,,$$

$$p = \frac{e}{2\pi}\int G_{1X}\Big|_{Z\to g_0(z)} d\theta = \frac{-e}{\Omega^2}\nabla_\perp \phi \,.$$

$$G_{1X} = \frac{1}{B}\times\nabla S_1 \,.$$



Lee (1983) used different method.
Friedman (1981) in different context.

Gyrocenter gauge theory for waves in magnetized plasmas

$F = F_0 + F_1$
$\boldsymbol{B}_0 = const.$

Pullback of current

$$\boldsymbol{j} = \int d^3v \boldsymbol{v} f(\boldsymbol{r},\boldsymbol{v},t) = \int d^6\bar{Z}\bar{\boldsymbol{V}}\bar{F}(\bar{Z},t)\delta(g_0^{-1}\bar{\boldsymbol{X}} - \boldsymbol{r})$$
$$= \int d^6\bar{Z}\bar{\boldsymbol{V}}[g_1^* F]\delta(g_0^{-1}\bar{\boldsymbol{X}} - \boldsymbol{r})$$

Lab coord.

0th order gyrocenter

Pullback of F

$$g_1^* F = F + L_G F = F - \frac{\boldsymbol{b}}{B_\parallel^\dagger} \times [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) + \boldsymbol{\nabla} S] \cdot \boldsymbol{\nabla} F - \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger} \frac{\partial S}{\partial u} \cdot \boldsymbol{\nabla} F$$
$$+ \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger} \cdot [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) + \boldsymbol{\nabla} S] \frac{\partial F}{\partial u_\parallel} + [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) \cdot \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + \frac{\partial S}{\partial \theta}] \frac{\partial F}{\partial \mu}$$
$$- [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) \cdot \frac{\partial \boldsymbol{\rho}_0}{\partial \mu} + \frac{\partial S}{\partial \mu}] \frac{\partial F}{\partial \theta}$$

Gyrocenter gauge theory for waves in magnetized plasmas

$$\frac{\partial F_1}{\partial t} + u\boldsymbol{b}\cdot\boldsymbol{\nabla}F_1 = \frac{1}{m}\boldsymbol{b}\cdot\boldsymbol{\nabla}\langle\psi_1\rangle\frac{\partial F_0}{\partial u}$$

$$\frac{\partial S}{\partial t} + \dot{\boldsymbol{X}}\cdot\frac{\partial S}{\partial \boldsymbol{X}} + u\frac{\partial S}{\partial u} + \dot{\theta}\frac{\partial S}{\partial \theta} = \tilde{\phi}(\boldsymbol{X}+\boldsymbol{\rho}_0,t) - \widetilde{\boldsymbol{v}\cdot\boldsymbol{A}}(\boldsymbol{X}+\boldsymbol{\rho}_0,t)$$

Gyrocenter gauge function

$$F_1 = \frac{-k_z}{(\omega - k_z u)}[J_0(\phi - uA_\parallel) + V_\perp J_1 A_y]\frac{\partial F_0}{\partial u}$$

$$S^* = S - \frac{H_1}{i(\omega - k_z u)}$$

$$S^* = \sum_{n=-\infty}^{n=\infty}\{\frac{I_n(\lambda)\,\mathrm{e}^{\mathrm{i}n\theta}}{i(n-\overline{\omega}+\overline{k_z u})}(\phi - uA_z) + \frac{nI_n(i\lambda)\,\mathrm{e}^{\mathrm{i}n\theta}}{-i\lambda(n-\overline{\omega}+\overline{k_z u})}V_\perp A_x + \frac{I_n'(i\lambda)\,\mathrm{e}^{\mathrm{i}n\theta}}{i(n-\overline{\omega}+\overline{k_z u})}V_\perp A_y\},$$

Solution

Pullback of current

$$\boldsymbol{j}_1 = \left\{ e \int (\boldsymbol{v}_\perp + u\boldsymbol{b})[g_1^*(F_0 + F_1)](Z)\delta(\boldsymbol{X} + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6 Z \right\}_1,$$

$$[g_1^*(F_0 + F_1)]_1 = F_1 + \boldsymbol{b} \cdot [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0) + \boldsymbol{\nabla} S] \frac{\partial F_0}{\partial u}$$

$$+ [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0) \cdot \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + \frac{\partial S}{\partial \theta}] \frac{\partial F_0}{\partial \mu}$$

$$\boldsymbol{j}_1 = e \int \delta(\boldsymbol{X} + \boldsymbol{\rho}_0 - \boldsymbol{r}) d^6 Z (\boldsymbol{v}_\perp + u\boldsymbol{b}) \left\{ A_z(\boldsymbol{X} + \boldsymbol{\rho}_0) \frac{\partial F_0}{\partial u} \right.$$

$$\left. + i k_z S^* \frac{\partial F_0}{\partial u} + [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0) \cdot \frac{\partial \rho_0}{\partial \xi} + \frac{\partial S^*}{\partial \theta}] \frac{\partial F_0}{\partial \mu} \right\}$$

$$= e \int d^3 V \; \mathrm{e}^{-\rho_0 \cdot \nabla} \; (\boldsymbol{v}_\perp + u_\parallel \boldsymbol{b}) \left\{ A_z(\boldsymbol{X} + \boldsymbol{\rho}_0) \frac{\partial F_0}{\partial u} \right.$$

$$\left. + i k_z S^* \frac{\partial F_0}{\partial u} + [\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{\rho}_0) \cdot \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + \frac{\partial S^*}{\partial \theta}] \frac{\partial F_0}{\partial \mu} \right\}_{X \mapsto r}$$

$$= -\frac{i\omega}{4\pi} \chi_p \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \\ \phi \end{pmatrix}$$

Gyrocenter gauge susceptibility – same as classical theory

$$\chi_p^{\parallel} = \frac{4\pi e^2}{i\omega m\Omega}\sum_{n=-\infty}^{n=\infty} 2\pi\int_0^{\infty} v_{\perp}dv_{\perp}\int_{-\infty}^{\infty} du\frac{1}{(n-\overline{\omega}+\overline{k_z u})}\frac{\partial F_0}{\partial u}$$

$$\times\begin{pmatrix} \frac{n^2J_n^2(\lambda)k_z}{\lambda^2 c}v_{\perp}^2 & \frac{inJ_n(\lambda)J_n^{'}(\lambda)k_z}{\lambda c}v_{\perp}^2 & \frac{-nJ_n^2(\lambda)(n-\overline{\omega})\Omega}{\lambda c}v_{\perp} & \frac{-nJ_n^2(\lambda)k_z\Omega}{k_x} \\ -\frac{inJ_n(\lambda)J_n^{'}(\lambda)k_z}{\lambda c}v_{\perp}^2 & \frac{J_n^{'2}(\lambda)k_z}{c}v_{\perp}^2 & \frac{J_n(\lambda)J_n^{'}(\lambda)(n-\overline{\omega})\Omega}{c}v_{\perp} & iJ_n(\lambda)J_n^{'}(\lambda)k_z v_{\perp} \\ \frac{nJ_n^2(\lambda)k_z}{\lambda c}v_{\perp V\parallel} & \frac{iJ_n(\lambda)J_n^{'}(\lambda)k_z}{c}v_{\perp V\parallel} & \frac{-J_n^2(\lambda)(n-\overline{\omega})\Omega}{c}u & -J_n^2(\lambda)k_z u \end{pmatrix},$$

$$\chi_p^{\perp} = \frac{4\pi e^2}{i\omega m\Omega}\sum_{n=-\infty}^{n=\infty} 2\pi\int_0^{\infty} v_{\perp}dv_{\perp}\int_{-\infty}^{\infty} du\frac{1}{(n-\overline{\omega}+\overline{k_z u})}\frac{\partial F_0}{\partial v_{\perp}}$$

$$\times\begin{pmatrix} \frac{n^2J_n^2(\lambda)(\omega-k_z u)}{\lambda^2 c}v_{\perp} & \frac{inJ_n(\lambda)J_n^{'}(\lambda)(\omega-k_z u)}{\lambda c}v_{\perp} & \frac{n^2J_n^2(\lambda)\Omega}{\lambda c}u & \frac{-n^2J_n^2(\lambda)\Omega}{\lambda} \\ \frac{-inJ_n(\lambda)J_n^{'}(\lambda)(\omega-k_z u)}{\lambda c}v_{\perp} & \frac{J_n^{'2}(\lambda)(\omega-k_z u)}{c}v_{\perp} & \frac{-inJ_n(\lambda)J_n^{'}(\lambda)\Omega}{c}u & inJ_n(\lambda)J_n^{'}(\lambda)\Omega \\ \frac{nJ_n^2(\lambda)(\omega-k_z u)}{\lambda c}u & \frac{iJ_n(\lambda)J_n^{'}(\lambda)(\omega-k_z u)}{c}u & \frac{nJ_n^2(\lambda)\Omega u^2}{cv_{\perp}} & \frac{-nJ_n^2(\lambda)\Omega u_{\parallel}}{v_{\perp}} \end{pmatrix}.$$

Gyrocenter gauge algorithm

Gyrokinetic eq.

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathbf{x}} \langle F \rangle + \frac{du}{dt} \frac{\partial \langle F \rangle}{\partial u} = 0\,,$$

Gyrocenter gauge

$$\frac{\partial S}{\partial t} + \dot{\boldsymbol{X}} \cdot \frac{\partial S}{\partial \boldsymbol{X}} + u \frac{\partial S}{\partial u} + \dot{\theta} \frac{\partial S}{\partial \theta} = \tilde{\phi}(\boldsymbol{X} + \boldsymbol{\rho}_0, t) - \widetilde{\boldsymbol{v} \cdot \boldsymbol{A}}(\boldsymbol{X} + \boldsymbol{\rho}_0, t)$$

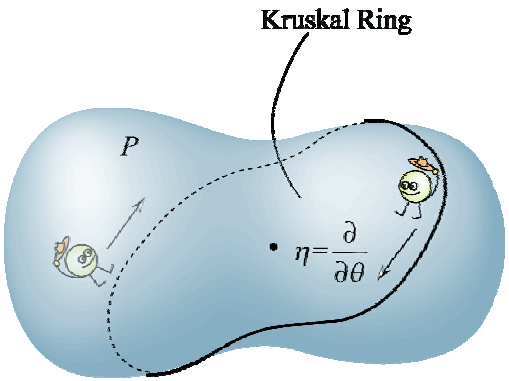
Pullback

$$f(z) = g^*[F(Z)] = F(g(z)).$$

Maxwell's

$$d * dA = 4\pi \int_{\pi^{-1}(x)} f\Omega$$

Gyrocenter gauge data structure – Kruskal ring



replace particle

Kruskal ring

- species
- X
- u
- μ
- $s[n]$

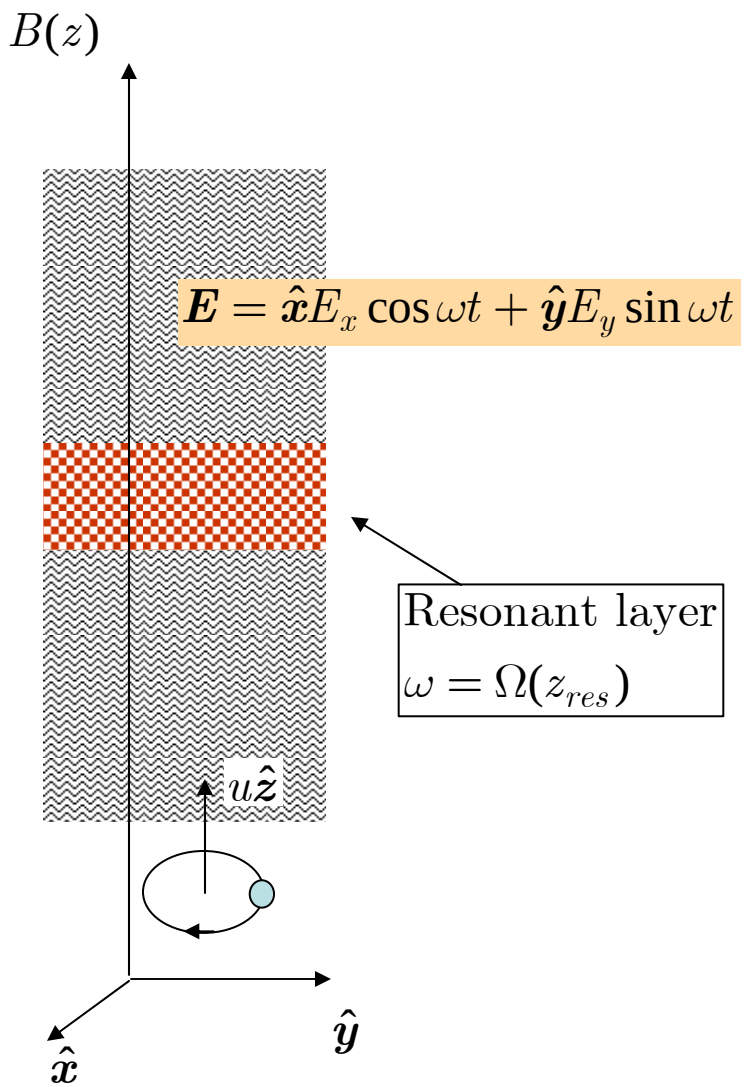
gyrocenter

SciDAC proposal

replace gyrophase

gyrocenter gauge

Example – stochastic resonant heating



Gyrocenter gauge

$$\frac{dS}{dt} = \tilde{\phi} \approx -\boldsymbol{\rho} \cdot \boldsymbol{E}$$

Gyrocenter

$$z = z_0 + ut, \quad \frac{d}{dt}\left(\mu + G_{1\mu}\right) = 0, \ \dot{\theta} = \Omega$$

$$\frac{d\mu}{dt} = -\frac{d}{dt} G_{1\mu} = -\frac{d}{dt}\frac{\partial S}{\partial \theta}$$

$$= -\frac{\partial}{\partial \theta}\frac{dS}{dt} = \frac{\partial \boldsymbol{\rho}}{\partial \theta} \cdot \boldsymbol{E} = \sqrt{\frac{2\mu}{B}}\hat{\boldsymbol{v}}_\perp \cdot \boldsymbol{E}$$

Stochastic resonant heating rate

$$\frac{1}{\sqrt{2\mu}}\frac{d\mu}{dt} = \left[-\frac{E_x}{\sqrt{B}}\sin\theta(t)\cos\omega t - \frac{E_y}{\sqrt{B}}\cos\theta(t)\sin\omega t\right]$$

$$\sqrt{2\mu} = \sqrt{2\mu}\,|_{t=0} + \int_{t=0}^{t}\left[-\frac{E_x}{\sqrt{B}}\sin\theta(t)\cos\omega t - \frac{E_y}{\sqrt{B}}\cos\theta(t)\sin\omega t\right]$$

$$\approx \sqrt{2\mu}\,|_{t=0} + \frac{\left(E_y - E_x\right)}{\sqrt{B}}\sin\left(\theta_{res} - \omega t_{res}\right)\sqrt{\frac{2\pi}{\Omega'}}\frac{\sqrt{2}}{2}$$

$$\int_{t=0}^{t}\sin\theta(t)\cos\omega t \approx -\sin\left(\theta_{res} - \omega t_{res}\right)\sqrt{\frac{2\pi}{\Omega'}}\cos\left(\frac{\pi}{4}\right)$$

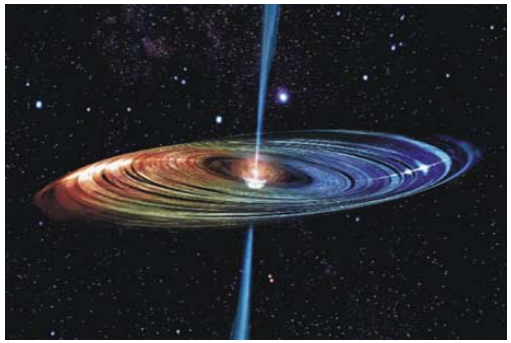
$$\int_{t=0}^{t}\cos\theta(t)\sin\omega t \approx +\sin\left(\theta_{res} - \omega t_{res}\right)\sqrt{\frac{2\pi}{\Omega'}}\cos\left(\frac{\pi}{4}\right)$$

Asymptotic for large Ω'

$$\Delta\left\langle\frac{v_\perp^2}{2}\right\rangle = \frac{\pi}{4}\frac{\left(E_y - E_x\right)^2}{\Omega'}$$

average over random phase

Conclusion – geometrically generalized Vlasov-Maxwell equations

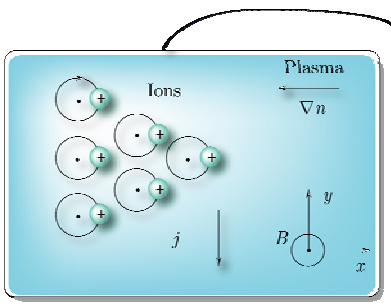


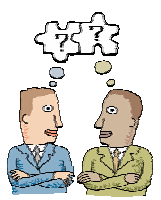
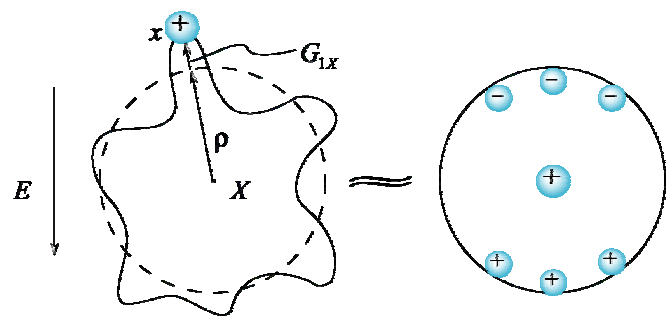
Kruskal ring

- species
- X
- u
- μ
- $s[n]$

Fundamental Theory = Efficient algorithm

Physics = Geometry





Plasma Physicists